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An iterative method for shakedown analysis

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Abstract

Shakedown analysis for elastic–perfect plastic structures is discussed and a fast incremental-iterative solution method is proposed, suitable for the FEM analyses of large structures.

The theoretical motivations of the proposed method are discussed in detail and an example of its implementation is described with reference to plane frame analysis.

Some numerical results are presented showing the numerical performances of the method.

Keywords: Plasticity; Shakedown analysis; Limit analysis; Nonlinear methods

1. Introduction

The elastic–plastic analysis has assumed a great importance also in civil engineering due to the diffusion of the semi-probabilistic approach to limit states, such as that established in Eurocodes (the European proposal codes), which allows the design of structures beyond the elastic limit.

When considering a single set of external loads, monotonically increasing with a load parameter, the safety factor of an elastic–plastic structure can be effectively evaluated by numerical implementations of the classical theorems of limit analysis [9–13] or, even more efficiently, by recovering the equilibrium path by means of path-following algorithms [14,15], based on the Riks' arc-length method [16] and well suited to be implemented in general purpose FEM codes (see [18,19] for an overview of these topics). It is known, however, that the collapse multiplier doesn't furnish a reliable safety index when the structure is subject to a combination of loads that can vary, cyclically or in a generic way, inside a given load's domain. In this case, in fact, other different failure modes are possible that also have to be prevented. A continuous increase in plastic deformations, along successive plastically admissible load cycles, could lead to a loss of the structure functionality (incremental collapse or ratchetting) or produce the collapse due to fatigue (alternating plasticity). Then, a further requirement has to be met, that the rising amount of plastic deformations be

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confined only to an initial phase after which the structure behavior, for any combination of load contained in the load's domain, is purely elastic. If this happens, we say that the structure shakes down to an elastic state or, simply, shakedown occurs.

The definition of conditions that imply shakedown of elastic–perfectly plastic structures has been widely studied, at least from a theoretical point of view, and shakedown theorems [20,36] currently represent one of most important achievements of the theory of plasticity [20]. Noticeable developments have been made that extend and generalize the shakedown theory. In particular a deeper insight into the extensions of the classical theory to material with hardening effects, finite displacements, thermal and dynamics effects can be found in the papers by Ceradini [21,22], Capurso [12], Gokhfeld e Cherniavsky [23], König e Maier [2], Ponter [24], Stein et al. [25], Yan and Nguyen [26] and Polizzotto et al. [27,28]. The reader can refer to the book by König [3] and to the reviews [1–6], for a general overview and historical details.

The use of these theoretical results suffers, still today, from the lack of efficient computational algorithms able to calculate the shakedown safety factor, that is, the maximum load amplifier that ensures shakedown, for large life-scale structures modeled using standard finite element formulations. As underlined by Groß-Weege [31] and by Zhang [30] and Janas and et al. [29], the proposed solution methods appear to be aimed more at academic purposes or to the analysis of specific problems than to be effectively implemented in FEM codes suitable for general technical applications. In effect most of the proposed numerical methods attempt to evaluate the shakedown multiplier directly from the shakedown theorems (direct methods). In this way, shakedown analysis is formulated as a standard constrained optimization problem to be solved by general methods of mathematical programming without exploiting the particular features of the structural problem. That implies a low computational efficiency in the overall solution process and practically prevents shakedown analysis from being easily implemented in commercial finite element codes.

More recently, an indirect approach has been proposed by Ponter et al. [24] that is based on the so-called elastic compensation methods. The shakedown safety factor is evaluated through an iterative sequence of pseudo-elastic solutions that produces a monotonically decreasing sequence of upper bounds that converges to the exact solution (obviously except for the approximations due to the finite element discretization). The method appears to be more efficient than direct methods and more suitable to a finite element implementation but it can still hardly be considered completely satisfactory from a computational point of view when compared with standard limit analysis algorithms. In fact it is characterized by a relatively slow convergence and requires a large number of complete elastic re-analyses (including the assemblage of the pseudo-elastic stiffness matrix and its Choleski decomposition).

A new method for the evaluation of the shakedown safety factor for elastic–perfectly plastic structures is presented in this paper. It is based on an iterative technique which has some analogies with the Riks path-following algorithm, currently used in elastic–plastic analysis to evaluate the equilibrium path of a structure, and offers the same characteristics of robustness, efficiency and computational requirements as that method. The full theoretical and computational aspects of the proposed method are discussed in detail, including the convergence proof of solution algorithms and numerical testings. The simple case of plane frames has been used here for exemplifying the implementation details and the numerical performances of the proposed method. Its extension to more complex structures can be considered quite straightforward.

The paper is organized as follows: Section 2 provides an introductory summary of classical shakedown theory; a reformulation of this theory, suitable for numerical analysis, is given in Section 3 and some preliminary results are provided; the proposed iterative method is described in Section 4 and its convergence properties are discussed; Sections 5 and 6 use the plane frame context to show an example of the actual implementation of the method; further comments and conclusions are given in Section 7.

2. Shakedown theory

In order to render the paper as self-contained as possible, we recall in this section a brief overview of the elastic–plastic theory and of the shakedown problem. The reader can refer to the general presentation by Koiter [20] or to the book by König [3], for a more detailed discussion.

2.1. Basic rules of the elastic–plastic theory

The basic rules of the classical elastic–plastic theory for perfect-plastic materials can be summarized as follows:

1. Stress and strain fields $\boldsymbol{\sigma}[x] \in \Re^d$ and $\boldsymbol{\varepsilon}[x] \in \Re^d$, $x \in B$ can be defined, d being the number of independent components of both $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ ($d = 6$ in 3D Cauchy model), and B is the body domain. The dot-product $\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon}$ is also defined for each $x \in B$ and we have
$$\int_B \bar{\boldsymbol{\sigma}}^{\mathrm{T}} \dot{\boldsymbol{\varepsilon}} \,\mathrm{d}v = 0 \tag{2.1}$$
for all self-equilibrated stress $\bar{\boldsymbol{\sigma}}$ and kinematically compatible strain increment field $\dot{\boldsymbol{\varepsilon}}$. Eq. (2.1) mutually defines the subspaces $\overline{\mathbb{S}}$ and $\mathbb{K}$ collecting all self-equilibrated stress fields and all kinematically compatible strain increments. We will generally omit, when referring to field relationships, explicit citations to $\forall x \in B$, which will be taken as implicit, for a lighter notation.
2. Stress $\boldsymbol{\sigma}$ is constrained to belong to the *admissible domain*
$$\mathbb{E} := \{\boldsymbol{\sigma} : f[\boldsymbol{\sigma}] \leqslant 0\}, \tag{2.2}$$
where $f[\boldsymbol{\sigma}]$ is a convex *yield function* in $\Re^d$, such that $f[\mathbf{0}] < 0$. Obviously, $\mathbb{E}$ will be closed and convex in $\Re^d$ and its boundary $\partial\mathbb{E}$ is characterized by $f[\boldsymbol{\sigma}] = 0$.
3. The total strain increment $\dot{\boldsymbol{\varepsilon}}$ can be subdivided into the elastic part $\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}$ and the plastic part $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$, and the two strain components are additive:
$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}. \tag{2.3}$$
4. The elastic component $\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}$ is linearly linked to the stress through the *elastic law*:
$$\dot{\boldsymbol{\sigma}} = \boldsymbol{E}\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} = \boldsymbol{E}(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}), \tag{2.4}$$
where $\boldsymbol{E}$ is the elastic tensor, symmetric and positive definite.
5. The plastic component $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ can be different from zero only if the stress $\boldsymbol{\sigma}$ belongs to $\partial\mathbb{E}$ and is defined by the *plastic flow rule*:
$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\mu}\boldsymbol{g}[\boldsymbol{\sigma}], \quad \dot{\mu} \geqslant 0, \ \ \dot{\mu}\dot{f} = 0, \ \ \dot{\mu} f = 0, \tag{2.5}$$
vector $\boldsymbol{g}$ being contained in the subdifferential [1] $\partial f[\boldsymbol{\sigma}]$ of the function f in $\boldsymbol{\sigma}$.

By these hypotheses, the following Drucker's conditions holds:
$$\left(\boldsymbol{\sigma}_y - \boldsymbol{\sigma}\right)^{\mathrm{T}} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \geqslant 0, \ \forall \boldsymbol{\sigma} \in \mathbb{E}, \tag{2.6}$$

[1] We briefly recall that the subdifferential of a function f in a point $\boldsymbol{\sigma}$ is the cone defined by the gradients of the function computed at all points adjacent to $\boldsymbol{\sigma}$ at infinitesimal distance and in all directions. It coincides with the classical gradient in regular points. The reader can refer to [34] for a general overview on convex analysis.

$\boldsymbol{\sigma}_y$ and $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ being related by the flow rule (2.5). Since $\boldsymbol{\sigma} = \mathbf{0}$ is internal to $\mathbb{E}$, by definition, we also have

$$\boldsymbol{\sigma}_y^{\mathrm{T}} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} > 0, \ \forall \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \neq 0. \tag{2.7}$$

2.2. Basic assumptions of shakedown analysis

We assume that the external load $\boldsymbol{p}[t]$ is expressed as a combination of basic loads $\boldsymbol{p}_i$, varying arbitrarily with time t but, in any case, belonging to the admissible *load domain*

$$\mathbb{P} := \left\{ \boldsymbol{p}[t] := \sum_{i=1}^{p} \alpha_i[t] \boldsymbol{p}_i : \alpha_i^{\min} \leqslant \alpha_i[t] \leqslant \alpha_i^{\max}, \ \forall t \right\}, \tag{2.8}$$

where the factors $\alpha_i[t]$ are contained in a polyhedron in $\Re^{\mathrm{p}}$ [27,29]. Such an assumption conforms to the external load description used in many civil engineering norm codes and derives from the fact that the real loads evolution is often unknown while, in some statistical way, the excursion of the minimum and maximum values for every basic load $\boldsymbol{p}_i$ is known or, at least, we can get meaningful reference values for the maximum or minimum of the load factors for a given life-time of the structure. Obviously $\mathbb{P}$ is closed and convex, by definition.

If denoting with $\boldsymbol{\sigma}_{ei}$ the elastic solution due to the single basic load $\boldsymbol{p}_i$ we can define the domain $\mathbb{S}_e$ of the elastic stresses $\boldsymbol{\sigma}_e[t]$ associated to $\boldsymbol{p}[t]$ in the form

$$\mathbb{S}_e := \left\{ \boldsymbol{\sigma}_e[t] := \sum_{i=1}^{p} \alpha_i[t] \boldsymbol{\sigma}_{ei} : \alpha_i^{\min} \leqslant \alpha_i[t] \leqslant \alpha_i^{\max}, \ \forall t \right\}. \tag{2.9}$$

Set $\mathbb{S}_e$, represents the envelope of the elastic stresses, and collects the local values of the elastic stresses produced, at different instants, by load paths $\boldsymbol{p}[t]$ contained in $\mathbb{P}$. Obviously $\mathbb{S}_e$ is also closed and convex.

It is worth mentioning that time is assumed here as an evolution variable since we always consider negligible the dynamic effects due to the external loads. However, our presentation can be easily generalized to the presence of dynamic effects (see Ceradini [21,22] and [27,28]).

We can now state the following definition:

Definition 2.1 (*Shakedown*). We say that a structure shakes down to an elastic state or, simply, shakedown occurs if, after an initial phase during which the occurrence and the accumulation of plastic strain increments are possible, the structural response, for every load path $\boldsymbol{p}[t] \in \mathbb{P}$, tends to be purely elastic and is characterized by a finite total plastic work. That is

$$\int_{t=0}^{\infty} \left\{ \int_B \boldsymbol{\sigma}[t]^{\mathrm{T}} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}[t] \, \mathrm{d}v \right\} \mathrm{d}t < \infty,$$

where $\boldsymbol{\sigma}[t]$ and $\boldsymbol{\varepsilon}^{\mathrm{p}}[t]$ are the stress and plastic strain produced during the loading process, a superposed dot indicates time differentiation and $t = 0$ refers to the initial virgin state ($\boldsymbol{\varepsilon}^{\mathrm{p}}[0] = 0$).

An obvious corollary of this definition is that shakedown implies the existence of, at least, one time independent stress field $\bar{\boldsymbol{\sigma}} \in \overline{\mathbb{S}}$ such that

$$f[\boldsymbol{\sigma}_e[t] + \bar{\boldsymbol{\sigma}}] \leqslant 0, \ \forall \boldsymbol{\sigma}_e[t] \in \mathbb{S}_e, \tag{2.10}$$

because, due to Eq. (2.7), we necessarily have $\lim_{t\to\infty} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}[t] = 0$, $\forall \boldsymbol{p}[t] \in \mathbb{P}$ and then, due to the uniqueness of the incremental elastic solution, $\lim_{t\to\infty} \dot{\boldsymbol{\sigma}}[t] - \dot{\boldsymbol{\sigma}}_e[t] = 0$, that is, $\lim_{t\to\infty} \boldsymbol{\sigma}[t] = \boldsymbol{\sigma}_e[t] + \bar{\boldsymbol{\sigma}}$, $\forall \boldsymbol{\sigma}_e[t] \in \mathbb{S}_e$. Furthermore, $\boldsymbol{\sigma}[t]$ and $\boldsymbol{\sigma}_e[t]$ being equilibrated by the same load $\boldsymbol{p}[t]$, by definition, their difference $\bar{\boldsymbol{\sigma}}$ is self-equilibrated.

2.3. Shakedown theorems

Sufficient and necessary conditions for shakedown are given in the classic Bleich–Melan's static theorem [35,36] and Koiter's kinematic theorem [37]. For the reader's convenience, we recall both theorems in some detail here.

Theorem 2.1 (Static theorem of shakedown). *Shakedown occurs if there exists a time-independent self-equilibrated stress field $\bar{\boldsymbol{\sigma}}$ such that*

$$\boldsymbol{\sigma}^*[t] = \boldsymbol{\sigma}_e[t] + \bar{\boldsymbol{\sigma}}, \quad f[\boldsymbol{\sigma}^*] < 0, \ \forall \boldsymbol{\sigma}_e[t] \in \mathbb{S}_e. \tag{2.11}$$

Proof. For all loading paths $\boldsymbol{p}[t] \in \mathbb{P}$ the evolution law for stress $\boldsymbol{\sigma}[t]$ and strain $\boldsymbol{\varepsilon}[t]$ acting in the structure can be expressed as:

$$\begin{cases} \boldsymbol{\sigma}[t] = \boldsymbol{\sigma}_e[t] + \Delta\boldsymbol{\sigma}[t], \\ \boldsymbol{\varepsilon}[t] = \boldsymbol{E}^{-1}(\boldsymbol{\sigma}_e[t] + \Delta\boldsymbol{\sigma}[t]) + \boldsymbol{\varepsilon}^{\mathrm{p}}[t]. \end{cases}$$

$\Delta\boldsymbol{\sigma}[t]$ being a self-equilibrated stress field and $\boldsymbol{\varepsilon}^{\mathrm{p}}[t]$ the plastic strain associated to $\boldsymbol{\sigma}[t]$. If the following positive function is introduced

$$\Psi[t] := \frac{1}{2}\int_B (\boldsymbol{\sigma} - \boldsymbol{\sigma}^*)^{\mathrm{T}} \mathbf{E}^{-1} (\boldsymbol{\sigma} - \boldsymbol{\sigma}^*)\,\mathrm{d}v \geqslant 0$$

by differentiating with respect to t and denoting by $\dot{\boldsymbol{\varepsilon}}_e[t] = \boldsymbol{E}^{-1}\dot{\boldsymbol{\sigma}}_e[t]$ the strain associated to $\dot{\boldsymbol{\sigma}}_e[t]$, we have

$$\dot{\Psi}[t] = \int_B (\boldsymbol{\sigma} - \boldsymbol{\sigma}^*)^{\mathrm{T}} \mathbf{E}^{-1} (\dot{\boldsymbol{\sigma}}[t] - \dot{\boldsymbol{\sigma}}^{\mathrm{e}})\,\mathrm{d}v = \int_B (\boldsymbol{\sigma} - \boldsymbol{\sigma}^*)^{\mathrm{T}} (\dot{\boldsymbol{\varepsilon}}[t] - \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}[t])\,\mathrm{d}v - \int_B (\boldsymbol{\sigma} - \boldsymbol{\sigma}^*)^{\mathrm{T}} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\,\mathrm{d}v.$$

Obviously, both $\dot{\boldsymbol{\varepsilon}}[t]$ and $\dot{\boldsymbol{\varepsilon}}_e[t]$ are kinematically compatible and $\boldsymbol{\sigma} - \boldsymbol{\sigma}^*$ is self-equilibrated. So the first integral in the right side of the equation is zero. Furthermore, $\boldsymbol{\sigma}[t] \in \mathbb{E}$ being associated by the flow rule (2.5) to $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ and $\boldsymbol{\sigma}^*$ being internal to $\mathbb{E}$, we can apply inequality (2.6) in strict form. We obtain

$$\dot{\Psi}[t] = -\int_B (\boldsymbol{\sigma} - \boldsymbol{\sigma}^*)^{\mathrm{T}} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\,\mathrm{d}v < 0 \quad \text{if } \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \neq 0.$$

So, function $\Psi[t]$ is both lower bounded and monotonically decreasing during a plastic process. To avoid contradiction, this implies that

$$\lim_{t\to\infty} \dot{\Psi}[t] = 0 \Rightarrow \lim_{t\to\infty} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = 0.$$

Furthermore, by assuming $v > 0$ small enough to satisfy

$$f[(1+v)\boldsymbol{\sigma}^*[t]] \leqslant 0,$$

we have, because of Eq. (2.6),

$$(\boldsymbol{\sigma}[t] - (1+v)\boldsymbol{\sigma}^*[t])^{\mathrm{T}} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}[t] \geqslant 0,$$

that is, integrating on B and remembering the definition of $\dot{\Psi}[t]$,

$$v\int_B \boldsymbol{\sigma}[t]^{\mathrm{T}} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\,\mathrm{d}v \leqslant (1+v)\int_B (\boldsymbol{\sigma}[t] - \boldsymbol{\sigma}^*[t])^{\mathrm{T}} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\,\mathrm{d}v \equiv -(1+v)\dot{\Psi}[t].$$

Then, integrating on the overall loading process, we obtain

$$\int_{t=0}^{\infty}\left\{\int_B \boldsymbol{\sigma}[t]^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}[t]\,\mathrm{d}v\right\}\mathrm{d}t \leqslant \frac{1+v}{v}(\Psi[0]-\Psi[\infty]) \leqslant \frac{1+v}{2v}\int_B \bar{\boldsymbol{\sigma}}^{\mathrm{T}}\boldsymbol{E}^{-1}\bar{\boldsymbol{\sigma}}\,\mathrm{d}v < \infty$$

because of $\boldsymbol{\varepsilon}^{\mathrm{p}}[0]=0$, i.e. $\boldsymbol{\sigma}[0]=\boldsymbol{\sigma}_e[0]$, by definition and $\Psi[\infty]\geqslant 0$. So shakedown condition is satisfied. □

Theorem 2.2 (Kinematical theorem of shakedown). *The structure does not shakedown if there exists a kinematical admissible strain field* $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}=\sum_k \dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}}\in\mathbb{K}$ *such that*

$$\exists\boldsymbol{\sigma}_{ek}\in\mathbb{S}_e : \int_B \sum_k (\boldsymbol{\sigma}_{ek}-\boldsymbol{\sigma}_{yk})^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}}\,\mathrm{d}v > 0, \tag{2.12}$$

where $\{\dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}}, k=1,2,\ldots\}$ *is a set of plastic strain increments and* $\boldsymbol{\sigma}_{yk}$ *are associated to* $\dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}}$ *through the flow rule* (2.5).

Proof. We will prove the statement by absurd. By assuming that shakedown occurs, due to Eq. (2.10), a self-equilibrated time-independent stress state $\bar{\boldsymbol{\sigma}}$ will exist such that

$$f[\boldsymbol{\sigma}_e+\bar{\boldsymbol{\sigma}}]\leqslant 0, \quad \forall\boldsymbol{\sigma}_e\in\mathbb{S}_e.$$

From the Drucker condition (2.6) we obtain:

$$(\boldsymbol{\sigma}_{yk}-\boldsymbol{\sigma}_{ek}-\bar{\boldsymbol{\sigma}})^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}}\geqslant 0, \quad \forall k \text{ and } \forall\boldsymbol{\sigma}_{ek}\in\mathbb{S}_e.$$

Then, by summing on k and integrating on the body volume B, we obtain:

$$\int_B \sum_k \left(\boldsymbol{\sigma}_{yk}-\boldsymbol{\sigma}_{ek}\right)^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}}\,\mathrm{d}v - \int_B \bar{\boldsymbol{\sigma}}\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\,\mathrm{d}v \geqslant 0.$$

The last integral being zero because $\bar{\boldsymbol{\sigma}}$ is self-equilibrated, the previous condition implies

$$\int_B \sum_k \left(\boldsymbol{\sigma}_{ek}-\boldsymbol{\sigma}_{yk}\right)^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}}\,\mathrm{d}v \leqslant 0, \quad \forall\boldsymbol{\sigma}_{ek}\in\mathbb{S}_e$$

which is absurd by being in contradiction with the theorem requirements. □

2.4. Shakedown safety factor

Structural safety with respect to shakedown can be evaluated by relating to the larger multiplier factor that can be used for amplifying the load domain $\mathbb{P}$ or, equivalently, the stress domain $\mathbb{S}_e$ allowing for the structure shakedown.

We call *shakedown safety factor* (or shakedown multiplier) λ_a this value and can define it more precisely, by referring to the amplified elastic stresses $\lambda\boldsymbol{\sigma}_e$, as the sup of *strictly safe* $\bar{\lambda}_s$ multipliers that satisfies the requirements of the static theorem. that is,

$$\exists\bar{\boldsymbol{\sigma}}\in\overline{\mathbb{S}} : f[\bar{\lambda}_s\boldsymbol{\sigma}_e+\bar{\boldsymbol{\sigma}}]<0, \quad \forall\boldsymbol{\sigma}_e\in\mathbb{S}_e. \tag{2.13}$$

Equivalently, it can be defined as the inf of *strictly unsafe* multipliers $\bar{\lambda}_u$ that satisfies the requirements of the kinematical theorem, that is

$$\exists\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} := \sum_k \dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}}\in\mathbb{K}\ \exists\boldsymbol{\sigma}_{ek}\in\mathbb{S}_e : \int_B \sum_k (\bar{\lambda}_u\boldsymbol{\sigma}_{ek}-\boldsymbol{\sigma}_{yk})^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}}\,\mathrm{d}v > 0, \tag{2.14}$$

$\boldsymbol{\sigma}_{yk}$ and $\dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}}$ being associated through the flow rule (2.5). The two definitions actually coincide, that is,

$$\lambda_a := \sup \bar{\lambda}_s = \inf \bar{\lambda}_u. \tag{2.15}$$

This can be easily proved by assuming λ_a be characterized by the condition

$$\lambda_a : \min_{\bar{\boldsymbol{\sigma}}} \left\{ \max_{\boldsymbol{\sigma}_e, x} f[\lambda_a \boldsymbol{\sigma}_e + \bar{\boldsymbol{\sigma}}] \right\} = 0, \quad \bar{\boldsymbol{\sigma}} \in \overline{\mathbb{S}},\ \boldsymbol{\sigma}_e \in \mathbb{S}_e,\ x \in B.$$

Due to convexity of $f[\boldsymbol{\sigma}]$ and the assumption $f[0] < \mathbf{0}$, we have

$$f[c\boldsymbol{\sigma}] \leqslant cf[\boldsymbol{\sigma}] + (1-c)f[\mathbf{0}] < 0 \ \text{ if } \ 0 \leqslant c < 1,\ f[\boldsymbol{\sigma}] \leqslant 0$$

and then we obtain

$$\min_{\bar{\boldsymbol{\sigma}} \in \overline{\mathbb{S}}} f\left[\bar{\lambda}_s \boldsymbol{\sigma}_e + \frac{\bar{\lambda}_s}{\lambda_a} \bar{\boldsymbol{\sigma}}\right] < 0, \quad \forall \boldsymbol{\sigma}_e \in \mathbb{S}_e \text{ if } \bar{\lambda}_s < \lambda_a.$$

Therefore, $\bar{\lambda}_s < \lambda_a$ is strictly safe according to definition (2.13). Conversely, due to Eq. (2.6), we have

$$\min_{\bar{\boldsymbol{\sigma}} \in \overline{\mathbb{S}}} \left\{ \max_{\boldsymbol{\sigma}_e \in \mathbb{S}_e} (\boldsymbol{\sigma}_y - \lambda_a \boldsymbol{\sigma}_e - \bar{\boldsymbol{\sigma}})^{\mathrm{T}} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \right\} \geqslant 0, \quad \forall \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}},$$

$\boldsymbol{\sigma}_y$ and $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ being associated by the flow rule (2.5) and the equal sign being attained for some $\boldsymbol{\sigma}_{yk}^* := \lambda_a \boldsymbol{\sigma}_{ek}^* + \bar{\boldsymbol{\sigma}}^*$ and $\dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}*}$, by definition. We can characterize this solution by the condition

$$\min_{\dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}}, \bar{\boldsymbol{\sigma}} \in \overline{\mathbb{S}}} \int_B \sum_k (\boldsymbol{\sigma}_{yk} - \lambda_a \boldsymbol{\sigma}_{ek}^* - \bar{\boldsymbol{\sigma}})^{\mathrm{T}} \dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}} \, \mathrm{d}v = 0,$$

summation being extended to all k. That implies

$$\int_B \delta \bar{\boldsymbol{\sigma}}^{\mathrm{T}} \sum_k \dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}*} \, \mathrm{d}v = 0, \quad \forall \delta \bar{\boldsymbol{\sigma}} \in \overline{\mathbb{S}},$$

that is, because of Eq. (2.1), $\sum_k \dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}*} \in \mathbb{K}$. Due to Eq. (2.7), we have furthermore

$$\int_B \sum_k \lambda_a \boldsymbol{\sigma}_{ek}^{*\mathrm{T}} \dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}*} \, \mathrm{d}v = \int_B \sum_k \boldsymbol{\sigma}_{yk}^{*\mathrm{T}} \dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}*} \, \mathrm{d}v > 0$$

and consequently

$$\int_B \sum_k (\bar{\lambda}_u \boldsymbol{\sigma}_{ek}^* - \boldsymbol{\sigma}_{yk}^*)^{\mathrm{T}} \dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}*} \, \mathrm{d}v > 0 \quad \text{if } \bar{\lambda}_u > \lambda_a.$$

Therefore $\bar{\lambda}_u > \lambda_a$ is strictly unsafe according to definition (2.14).

In the sequel it will be convenient to refer to the *safe* multipliers λ_s and *unsafe* multipliers λ_u defined through Eqs. (2.13) and (2.14) by relaxing the "$<$" and "$>$" conditions into "$\leqslant$" and "$\geqslant$" ones. Obviously we obtain

$$\lambda_a = \max \lambda_s = \min \lambda_u.$$

2.5. Some comments

It is worth noting that the static shakedown theorem is a generalization of the static limit analysis theorem for a combination of independent external loads varying in an admissible domain $\mathbb{P}$. In particular the static limit analysis theorem is a special case of the static shakedown theorem for a single external load

($p = 1,\ 0 \leqslant \alpha_1 \leqslant \lambda$). Analogously, the kinematical shakedown theorem is a generalization of the kinematical theorem of the limit analysis and the latter can be considered its specialization for a single load. As in limit analysis, a constant self-equilibrated prestress is without influence on the shakedown multiplier.

Also note that static shakedown theorem assures the elastic multiplier

$$\lambda_e = \max \lambda : f[\lambda \boldsymbol{\sigma}_e] \leqslant 0, \quad \forall \boldsymbol{\sigma}_e \in \mathbb{S}_e$$

to be smaller (or, at least not larger) than the shakedown multiplier λ_a. In fact, by adding to the elastic stresses $\lambda \boldsymbol{\sigma}_e[t]$ a constant (zero) self-equilibrated stress, we are able to satisfy the theorem requirements. Conversely, the shakedown kinematic theorem assures that, for each load combination $\boldsymbol{p} \in \mathbb{P}$, the collapse multiplier

$$\lambda_c = \min \lambda : \int_B (\boldsymbol{\sigma}_y - \lambda \boldsymbol{\sigma}_e)^{\mathrm{T}} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \, \mathrm{d}v = 0, \quad \forall \boldsymbol{\sigma}_e \in \mathbb{S}_e,\ \forall \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \in \mathbb{K}$$

is larger (or, at least, not smaller) than λ_a. The shakedowm multiplier is then bounded as follows:

$$\lambda_e \leqslant \lambda_a \leqslant \lambda_c.$$

We can mention that the previous statements provide rational motivations for the use of the elastic multiplier in practical structural design. In fact in this way it is possible to surpass the shakedown analysis while banally fulfilling its requirements. However, it has to be considered that an explicit shakedown analysis is necessary when the design is based on nonlinear methods, procedures based on the evaluation of the collapse multiplier being unsafe.

3. A shakedown formulation suited to FEM analysis

A reformulation of the shakedown problem, suitable for FEM implementation, is presented in this section and some preliminary results are obtained that will be useful for discussing the convergence properties of the proposed solution method.

3.1. Shakedown admissible domain

It is convenient to introduce the *shakedown yield function* defined as

$$f_s[\boldsymbol{\sigma}, \lambda] := \max_{\sigma_e \in \mathbb{S}_e} \{ f[\lambda \boldsymbol{\sigma}_e + \boldsymbol{\sigma}] \} \tag{3.1}$$

and the related *shakedown admissible domain*

$$\mathbb{E}_s[\lambda] := \{ \boldsymbol{\sigma} : f_s[\boldsymbol{\sigma}, \lambda] \leqslant 0 \} \tag{3.2}$$

that represents the set of all possible translations $\boldsymbol{\sigma}$ of domain $\lambda \mathbb{S}_e$ within $\mathbb{E}$ (see Fig. 1). Obviously, due to the assumption $f[\mathbf{0}] < 0$, $\mathbb{E}_s[\lambda]$ is not void for sufficiently small positive values of λ. Furthermore, since both $\mathbb{S}$ and $\mathbb{E}$ are closed and convex, so $\mathbb{E}_s[\lambda]$ it is. We also have:

$$\mathbb{E}_s[\lambda_1] \neq \emptyset, \quad \mathbb{E}_s[\lambda_1] \text{ internal to } \mathbb{E}_s[\lambda_2] \quad \text{if} \quad \lambda_2 < \lambda_1 < \bar{\lambda} := \sup\{\lambda : \mathbb{E}_s[\lambda] \neq \emptyset\}. \tag{3.3}$$

Definition (3.2) implies the equivalence between the following statements

$$(\lambda \boldsymbol{\sigma}_e + \boldsymbol{\sigma}) \in \mathbb{E} \Longleftrightarrow \boldsymbol{\sigma} \in \mathbb{E}_s[\lambda].$$

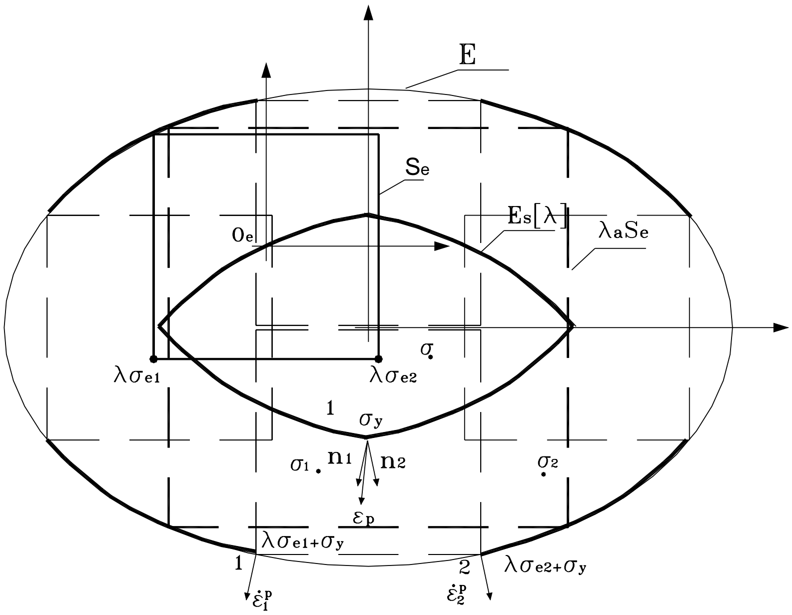

Fig. 1. Elastic domains $\mathbb{E}$ and $\mathbb{E}_s[\lambda]$.

Then, for $\lambda \leqslant \bar{\lambda}$ and $\boldsymbol{\sigma}_y \in \partial\mathbb{E}_s[\lambda]$, one or more tension fields $\boldsymbol{\sigma}_{yk} = \lambda\boldsymbol{\sigma}_{ek} + \boldsymbol{\sigma}_y \in \partial\mathbb{E}$ can be associated to $\boldsymbol{\sigma}_y$, as shown in Fig. 1. We have, from condition (2.6)

$$(\boldsymbol{\sigma}_y - \boldsymbol{\sigma})^{\mathrm{T}}\boldsymbol{\varepsilon}^{\mathrm{p}} = \sum_k (\boldsymbol{\sigma}_{yk} - \boldsymbol{\sigma}_k)^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}} \geqslant 0, \quad \forall \boldsymbol{\sigma} \in \mathbb{E}_s[\lambda], \tag{3.4}$$

where $\boldsymbol{\sigma}_k := \lambda\boldsymbol{\sigma}_{ek} + \boldsymbol{\sigma} \in \mathbb{E}$, $\dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}}$ are the plastic strains associated to $\boldsymbol{\sigma}_{yk}$, and $\boldsymbol{\varepsilon}^{\mathrm{p}}$, defined by the combination

$$\boldsymbol{\varepsilon}^{\mathrm{p}} := \sum_k \dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}} \tag{3.5}$$

is a plastic strain increment we can associate to $\boldsymbol{\sigma}_y$. Condition (3.4) implies the convexity of function $f_s[\boldsymbol{\sigma}, \lambda]$ and a normality rule between $\boldsymbol{\sigma}_y$ and $\boldsymbol{\varepsilon}^{\mathrm{p}}$:

$$\boldsymbol{\varepsilon}^{\mathrm{p}} = \mu\boldsymbol{g}, \quad \boldsymbol{g} \in \partial f_s[\boldsymbol{\sigma}_y; \lambda], \quad \mu \geqslant 0, \quad \mu f_s = \mu\dot{f}_s = 0. \tag{3.6}$$

3.2. The shakedown problem

Using the previous definitions, we can derive a simple characterization for the shakedown safety factor. We obtain, from static theorem and Eq. (3.1),

$$\lambda_s \leqslant \lambda_a \quad \text{if } \exists\bar{\boldsymbol{\sigma}} \in \overline{\mathbb{S}} : f_s[\bar{\boldsymbol{\sigma}}, \lambda_s] \leqslant 0 \tag{3.7}$$

and, from kinematic theorem and the assumptions $\boldsymbol{\sigma}_y = \boldsymbol{\sigma}_{yk} - \lambda\boldsymbol{\sigma}_{ek}$ and $\boldsymbol{\varepsilon}^{\mathrm{p}} = \sum_k \dot{\boldsymbol{\varepsilon}}_k^{\mathrm{p}}$,

$$\lambda_u \geqslant \lambda_a \quad \text{if } \exists\boldsymbol{\varepsilon}^{\mathrm{p}} \in \mathbb{K} : \int_B \boldsymbol{\sigma}_y^{\mathrm{T}}\boldsymbol{\varepsilon}^{\mathrm{p}}\,\mathrm{d}v \leqslant 0, \quad \boldsymbol{\varepsilon}^{\mathrm{p}} \neq 0. \tag{3.8}$$

We can now define the shakedown problem:

Definition 3.1 (*Shakedown problem*). Determine the shakedown safety factor

$$\lambda_a := \max \lambda_s = \min \lambda_u,$$

λ_s and λ_u being the "safe" and "unsafe" multipliers defined according to Eqs. (3.7) and (3.8).

3.3. Return mapping to the admissible domain

Starting from a given amplifier $\lambda \leqslant \bar{\lambda}$ and a stress $\boldsymbol{\sigma}^*$, not necessarily contained in the admissible domain $\mathbb{E}_s[\lambda]$, we can define the following process, which we call *return mapping* to the admissible domain (Fig. 2)

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_a[\boldsymbol{\sigma}^*, \lambda] := \boldsymbol{\sigma}^* - \boldsymbol{E}\boldsymbol{\varepsilon}^{\mathrm{p}}, \quad \boldsymbol{\sigma} \in \mathbb{E}_s[\lambda] \tag{3.9}$$

that allows us to determine an associated stress $\boldsymbol{\sigma}$, contained in $\mathbb{E}_s[\lambda]$ and related to $\boldsymbol{\varepsilon}^{\mathrm{p}}$ by the flow rule condition (3.6) we can rewrite

$$\boldsymbol{\varepsilon}^{\mathrm{p}} = \mu \boldsymbol{g}, \quad \boldsymbol{g} \in \partial f_s[\boldsymbol{\sigma}; \lambda], \quad \mu : \begin{cases} = 0, & \text{if } f_s[\boldsymbol{\sigma}^*, \lambda] < 0, \\ \geqslant 0, f_s[\boldsymbol{\sigma}, \lambda] = 0, & \text{if } f_s[\boldsymbol{\sigma}^*, \lambda] \geqslant 0. \end{cases} \tag{3.10}$$

We know (see [18]) that the reduction scheme (3.9) and (3.10) corresponds to the so-called "return mapping by closest-point projection" and can be conveniently obtained by minimizing the Haar–Kármán function, that is, by the condition:

$$\phi[\boldsymbol{\sigma} - \boldsymbol{\sigma}^*] := \tfrac{1}{2}\{(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*)^{\mathrm{T}} \boldsymbol{E}^{-1} (\boldsymbol{\sigma} - \boldsymbol{\sigma}^*)\} = \min, \quad \forall \boldsymbol{\sigma} \in \mathbb{E}_s[\lambda], \tag{3.11}$$

which will prove to be better suited for the numerical implementation. Obviously, defined as the minimum of a strictly convex function on a convex domain, $\boldsymbol{\sigma}_a$ is a single valued function of both $\boldsymbol{\varepsilon}$ and λ.

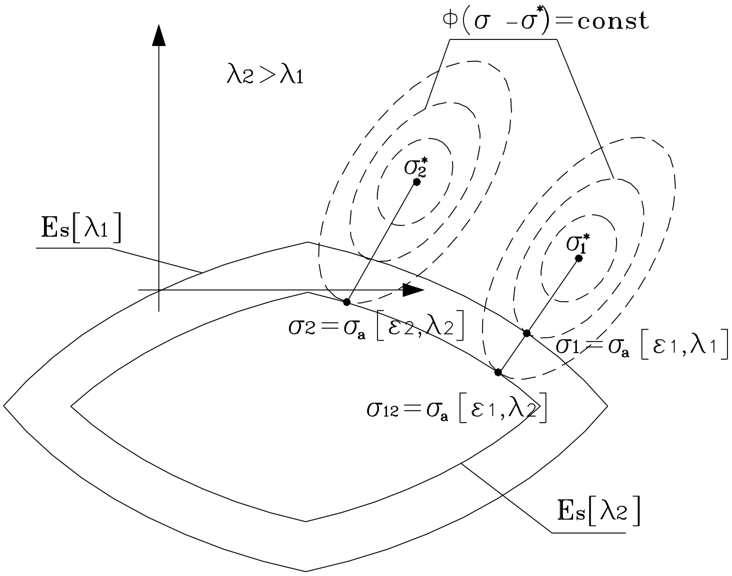

Fig. 2. Elastic domain $\mathbb{E}_s[\lambda]$ for $\lambda_2 > \lambda_1$.

3.4. Basic properties of the return mapping scheme

In order to better understand the solution algorithm, that will be proposed in the next section, and prove its convergence properties, it is convenient to state some basic characteristics of the return mapping scheme (3.9) and (3.10).

Let $\{\lambda < \bar{\lambda}, \boldsymbol{\sigma}_0\}$ be an initial state and let $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}_0 + \boldsymbol{E}\boldsymbol{\varepsilon}$, we can express $\boldsymbol{\sigma}$ as a function of $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}, \lambda] := \boldsymbol{\sigma}_a[\boldsymbol{\sigma}_0 + \boldsymbol{E}\boldsymbol{\varepsilon}, \lambda] = \boldsymbol{\sigma}_0 + \boldsymbol{E}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}). \tag{3.12}$$

Noticeable properties of this function are given by the following lemmas:

Lemma 3.1. *Function $\boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}, \lambda]$ is directionally differentiable with respect to both $\boldsymbol{\varepsilon}$ and λ. Furthermore there exists a convex potential function $\Psi[\boldsymbol{\varepsilon}, \lambda]$ such that*

$$\boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}, \lambda] = \frac{\partial \Psi[\boldsymbol{\varepsilon}, \lambda]}{\partial \boldsymbol{\varepsilon}}$$

so the tangent operator

$$\boldsymbol{E}_t[\boldsymbol{\varepsilon}, \lambda] := \frac{\partial \boldsymbol{\sigma}_a}{\partial \boldsymbol{\varepsilon}} = \frac{\partial^2 \Psi}{\partial \boldsymbol{\varepsilon}^2} \tag{3.13}$$

is self-adjoint: $\boldsymbol{E}_t = \boldsymbol{E}_t^{\mathrm{T}}$.

Proof. This is a classical result in incremental plasticity (e.g. see [38,39]) and derives directly from the definition of $\boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}, \lambda]$ through Haar–Kármán condition (3.11). It can be easily proven by considering that, due to the assumption on $f[\boldsymbol{\sigma}]$ and definition (3.1), function $f_s[\boldsymbol{\sigma}, \lambda]$ is a bounded, single valued continuous mapping $\Re^{d+1} \to \Re$, so it is directionally differentiable. Therefore, for any path $\{\boldsymbol{\varepsilon} + t\hat{\boldsymbol{\varepsilon}}, \lambda + t\hat{\lambda}\}$, $t \geqslant 0$, we can define the directional derivatives $\boldsymbol{\sigma}_t[\boldsymbol{\varepsilon}, \lambda; \hat{\lambda}]$ and $\boldsymbol{E}_t[\boldsymbol{\varepsilon}, \lambda; \hat{\boldsymbol{\varepsilon}}]$ such that

$$\mathrm{d}\boldsymbol{\sigma} = \boldsymbol{E}_t \, \mathrm{d}\boldsymbol{\varepsilon} + \boldsymbol{\sigma}_t \, \mathrm{d}\lambda.$$

The existence of the potential function Ψ is assured if

$$\oint_C \boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}, \lambda]^{\mathrm{T}} \, \mathrm{d}\boldsymbol{\varepsilon} = 0$$

along any closed curves C in the $\boldsymbol{\varepsilon}$ space. Letting $\mathrm{d}\boldsymbol{\varepsilon} = \boldsymbol{E}^{-1} \mathrm{d}\boldsymbol{\sigma}_a + \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}}$ we can write

$$\boldsymbol{\sigma}_a^{\mathrm{T}} \, \mathrm{d}\boldsymbol{\varepsilon} = \boldsymbol{\sigma}_a^{\mathrm{T}} \boldsymbol{E}^{-1} \, \mathrm{d}\boldsymbol{\sigma}_a + \boldsymbol{\sigma}_a^{\mathrm{T}} \, \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} = \mathrm{d}\left(\tfrac{1}{2}\boldsymbol{\sigma}_a^{\mathrm{T}} \boldsymbol{E}^{-1} \boldsymbol{\sigma}_a + \boldsymbol{\sigma}_a^{\mathrm{T}} \boldsymbol{\varepsilon}^{\mathrm{p}}\right) - \mathrm{d}\boldsymbol{\sigma}_a^{\mathrm{T}} \boldsymbol{\varepsilon}^{\mathrm{p}}$$

that implies

$$\oint_C \boldsymbol{\sigma}_a^{\mathrm{T}} \, \mathrm{d}\boldsymbol{\varepsilon} = -\oint_C \mathrm{d}\boldsymbol{\sigma}_a^{\mathrm{T}} \boldsymbol{\varepsilon}^{\mathrm{p}}.$$

Now, due to the flow rule (3.10), we have $\boldsymbol{\varepsilon}^{\mathrm{p}} = 0$ if $\boldsymbol{\sigma}_a$ is internal to $\mathbb{E}_s[\lambda]$; we have $\mathrm{d}\boldsymbol{\sigma}_a = 0$ if $\boldsymbol{\sigma}_a$ corresponds to a corner point of $\partial\mathbb{E}_s[\lambda]$ and $\boldsymbol{\varepsilon}^{\mathrm{p}}$ is internal to the cone of normals, and $\mathrm{d}\boldsymbol{\sigma}_a^{\mathrm{T}} \boldsymbol{\varepsilon}^{\mathrm{p}} = 0$ in other cases. Therefore, we obtain $\oint_c \mathrm{d}\boldsymbol{\sigma}_a^{\mathrm{T}} \boldsymbol{\varepsilon}^{\mathrm{p}} \equiv 0$, by proving the second part of the statement. □

Lemma 3.2. *Let $\lambda < \bar{\lambda}$, $\boldsymbol{\sigma}_0$ be an initial stress and $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ be stresses obtained from the strain increments $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ through return mapping* (3.11):

$$\begin{cases} \boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}_1, \lambda] = \boldsymbol{\sigma}_0 + \boldsymbol{E}(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_1^{\mathrm{p}}), \\ \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}_2, \lambda] = \boldsymbol{\sigma}_0 + \boldsymbol{E}(\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_2^{\mathrm{p}}), \end{cases}$$

where $\boldsymbol{\varepsilon}_1^{\mathrm{p}}$ *and* $\boldsymbol{\varepsilon}_2^{\mathrm{p}}$ *are related to* $\boldsymbol{\sigma}_1$ *and* $\boldsymbol{\sigma}_2$ *through the flow rule* (3.6). *The following conditions hold*:

$$0 \leqslant (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^{\mathrm{T}}(\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1) \leqslant (\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1)^{\mathrm{T}}\boldsymbol{E}(\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1)$$

and the occurrence of $(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^{\mathrm{T}}(\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1) = 0$ *implies*

$$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2.$$

Proof. Druker's condition (2.6) provides

$$(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^{\mathrm{T}}\boldsymbol{\varepsilon}_2^{\mathrm{p}} \geqslant 0, \qquad (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^{\mathrm{T}}\boldsymbol{\varepsilon}_1^{\mathrm{p}} \geqslant 0,$$

then, by their combination, we obtain

$$(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^{\mathrm{T}}(\boldsymbol{\varepsilon}_2^{\mathrm{p}} - \boldsymbol{\varepsilon}_1^{\mathrm{p}}) \geqslant 0.$$

From Eq. (3.2) we also have

$$(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1) = \boldsymbol{E}(\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1) - \boldsymbol{E}(\boldsymbol{\varepsilon}_2^{\mathrm{p}} - \boldsymbol{\varepsilon}_1^{\mathrm{p}}).$$

We finally obtain:

$$\begin{aligned}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^{\mathrm{T}}(\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1) &= (\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1)^{\mathrm{T}}\boldsymbol{E}(\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1) - (\boldsymbol{\varepsilon}_2^{\mathrm{p}} - \boldsymbol{\varepsilon}_1^{\mathrm{p}})^{\mathrm{T}}\boldsymbol{E}(\boldsymbol{\varepsilon}_2^{\mathrm{p}} - \boldsymbol{\varepsilon}_1^{\mathrm{p}}) - (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^{\mathrm{T}}(\boldsymbol{\varepsilon}_2^{\mathrm{p}} - \boldsymbol{\varepsilon}_1^{\mathrm{p}}) \\ &\leqslant (\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1)^{\mathrm{T}}\boldsymbol{E}(\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1) - (\boldsymbol{\varepsilon}_2^{\mathrm{p}} - \boldsymbol{\varepsilon}_1^{\mathrm{p}})^{\mathrm{T}}\boldsymbol{E}(\boldsymbol{\varepsilon}_2^{\mathrm{p}} - \boldsymbol{\varepsilon}_1^{\mathrm{p}}) \leqslant (\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1)^{\mathrm{T}}\boldsymbol{E}(\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1), \\ (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^{\mathrm{T}}(\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1) &= (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^{\mathrm{T}}\boldsymbol{E}^{-1}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1) + (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^{\mathrm{T}}(\boldsymbol{\varepsilon}_2^{\mathrm{p}} - \boldsymbol{\varepsilon}_1^{\mathrm{p}}) \geqslant (\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)^{\mathrm{T}}\boldsymbol{E}^{-1}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1)\end{aligned}$$

that proves the lemma statement. □

Lemma 3.3. *Matrix* $\boldsymbol{E}_t[\boldsymbol{\varepsilon}, \lambda; \boldsymbol{\varepsilon}]$ *satisfies the conditions*

$$0 \leqslant \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{E}_t\boldsymbol{\varepsilon} \leqslant \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{E}\boldsymbol{\varepsilon}, \quad \forall \boldsymbol{\varepsilon}.$$

Proof. The proof derives directly from Lemma 3.2 by taking $\boldsymbol{\varepsilon} := \boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1$ for $\boldsymbol{\varepsilon}_2 \to \boldsymbol{\varepsilon}_1$. □

Lemma 3.4. *Let* $\boldsymbol{\sigma}_1 := \boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}_1, \lambda_1]$ *and* $\boldsymbol{\sigma}_2 := \boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}_2, \lambda_2]$ *we can write their difference in the form*

$$\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_s[\boldsymbol{\varepsilon}_1, \lambda_1, \lambda_2](\lambda_2 - \lambda_1) + \boldsymbol{E}_s[\lambda_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2](\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1).$$

Furthermore, we have

$$\boldsymbol{E}_s = \boldsymbol{E}_s^{\mathrm{T}}, \qquad 0 \leqslant \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{E}_s\boldsymbol{\varepsilon} \leqslant \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{E}\boldsymbol{\varepsilon}, \quad \forall \boldsymbol{\varepsilon}$$

and

$$\boldsymbol{\sigma}_s[\boldsymbol{\varepsilon}_1, \lambda_1, \lambda_2]^{\mathrm{T}}\boldsymbol{\varepsilon}^{\mathrm{p}} < 0 \quad \text{if } \boldsymbol{\varepsilon}^{\mathrm{p}} := \left\{ \begin{matrix} \boldsymbol{\varepsilon}_{11}^{\mathrm{p}} & \text{if } \lambda_1 \leqslant \lambda_2 \\ \boldsymbol{\varepsilon}_{12}^{\mathrm{p}} & \text{if } \lambda_2 < \lambda_1 \end{matrix} \right\} \neq 0,$$

where $\boldsymbol{\varepsilon}_{11}^{\mathrm{p}}$ *and* $\boldsymbol{\varepsilon}_{12}^{\mathrm{p}}$ *are the plastic strain associated to* $\boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}_1, \lambda_1]$ *and* $\boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}_1, \lambda_2]$, *respectively*.

Proof. The first part of the statement is obtained directly by defining $\boldsymbol{E}_s$ and $\boldsymbol{\sigma}_s$ by the secant ratios

$$\boldsymbol{\sigma}_s[\boldsymbol{\varepsilon}_1, \lambda_1, \lambda_2] := \int_0^1 \frac{\partial \boldsymbol{\sigma}[\boldsymbol{\varepsilon}_1, \lambda[t]]}{\partial \lambda}\,\mathrm{d}t = \int_0^1 \boldsymbol{\sigma}_t[\boldsymbol{\varepsilon}_1, \lambda[t]]\,\mathrm{d}t, \quad \lambda[t] := t\lambda_1 + (1-t)\lambda_2,$$

$$\boldsymbol{E}_s[\lambda_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2] := \int_0^1 \frac{\partial \boldsymbol{\sigma}[\boldsymbol{\varepsilon}[t], \lambda_2]}{\partial \boldsymbol{\varepsilon}}\,\mathrm{d}t = \int_0^1 \boldsymbol{E}_t[\boldsymbol{\varepsilon}[t], \lambda_2]\,\mathrm{d}t, \quad \boldsymbol{\varepsilon}[t] := t\boldsymbol{\varepsilon}_1 + (1-t)\boldsymbol{\varepsilon}_2.$$

The second part of the lemma is simply obtained by observing that $\boldsymbol{E}_s$ is obtained as an average, on the segment $t \in [0 \cdots 1]$, of matrix $\boldsymbol{E}_t$ and recalling Lemmas 3.1 and 3.3. Finally, the third part is obtained by observing that, due to Eqs. (2.6) and (3.3), we have

$$(\boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}_1, \lambda_1] - \boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}_1, \lambda_2])^{\mathrm{T}} \boldsymbol{\varepsilon}_{11}^{\mathrm{p}} > 0 \quad \text{if } \lambda_1 < \lambda_2, \ \boldsymbol{\varepsilon}_{11}^{\mathrm{p}} \neq 0,$$

$$(\boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}_1, \lambda_2] - \boldsymbol{\sigma}_a[\boldsymbol{\varepsilon}_1, \lambda_1])^{\mathrm{T}} \boldsymbol{\varepsilon}_{12}^{\mathrm{p}} > 0 \quad \text{if } \lambda_2 < \lambda_1, \ \boldsymbol{\varepsilon}_{12}^{\mathrm{p}} \neq 0$$

and, for $\lambda_2 \to \lambda_1$ we obtain, by continuity,

$$\boldsymbol{\sigma}_s[\boldsymbol{\varepsilon}_1, \lambda_1, \lambda_1] \boldsymbol{\varepsilon}_{11}^{\mathrm{p}} < 0 \quad \text{if } \boldsymbol{\varepsilon}_{11}^{\mathrm{p}} \neq 0. \qquad \square$$

4. An iterative algorithm for shakedown analysis

A solution method, suitable to FEM analysis, is proposed in this section for the shakedown problem 3.1. The method is based on an incremental-iterative process producing a sequence $\lambda^{(k)}$, $k = 1, 2, \ldots$ of multipliers which, apart from the errors implied by the discretization, are safe according to Eq. (3.7) and converge monotonously to the shakedown safety factor λ_a.

4.1. Finite element discretization

Assuming the structure has already been modeled by a standard FEM discretization, let $\boldsymbol{u} \in \Re^N$ be the vector collecting all free nodal displacements, $\boldsymbol{u}_0$ its initial value and $\boldsymbol{\varepsilon}[\boldsymbol{u}] := \boldsymbol{D}(\boldsymbol{u} - \boldsymbol{u}_0)$ the (kinematically compatible) local strains associated to the displacement increment $\boldsymbol{u} - \boldsymbol{u}_0$ through compatibility matrix $\boldsymbol{D}[x]$. Assuming $\lambda \leqslant \bar{\lambda}$ we can define the internal force vector $\boldsymbol{s}[\boldsymbol{u}, \lambda] \in \Re^N$ expressing the structural response consequent to the nodal displacements $\boldsymbol{u}$:

$$\boldsymbol{s}[\boldsymbol{u}, \lambda] := \int_B \boldsymbol{D}^{\mathrm{T}} \boldsymbol{\sigma}[\boldsymbol{u}, \lambda] \, \mathrm{d}v, \qquad \boldsymbol{\sigma}[\boldsymbol{u}, \lambda] := \boldsymbol{\sigma}_a[\boldsymbol{\sigma}_0 + \boldsymbol{E}\boldsymbol{\varepsilon}, \lambda] \tag{4.1}$$

and the symmetric, positive definite elastic stiffness matrix $\boldsymbol{K}_e \in \Re^N \times \Re^N$

$$\boldsymbol{K}_e := \int_B \boldsymbol{D}^{\mathrm{T}} \boldsymbol{E} \boldsymbol{D} \, \mathrm{d}v, \qquad \boldsymbol{K}_e = \boldsymbol{K}_e^{\mathrm{T}} > 0 \tag{4.2}$$

by the energy identities:

$$\delta \boldsymbol{u}^{\mathrm{T}} \boldsymbol{s}[\boldsymbol{u}, \lambda] \equiv \int_B \delta \boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{\sigma}[\boldsymbol{u}, \lambda] \, \mathrm{d}v, \quad \delta \boldsymbol{u}^{\mathrm{T}} \boldsymbol{K}_e \, \delta \boldsymbol{u} \equiv \int_B \delta \boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{E} \, \delta \boldsymbol{\varepsilon} \, \mathrm{d}v, \qquad \forall \delta \boldsymbol{u}, \ \delta \boldsymbol{\varepsilon} := \boldsymbol{D} \, \delta \boldsymbol{u}, \tag{4.3}$$

$\boldsymbol{E}$ being the elastic matrix, linking the strain vector $\boldsymbol{\varepsilon}$ to the stress vector $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_0$ the initial stress state. Note that $\boldsymbol{\sigma}[\boldsymbol{u}, \lambda] \in \mathbb{E}_s[\lambda]$, by definition. According with Eq. (2.1), self-equilibrated stresses are characterized, for the discrete model, by the condition

$$\boldsymbol{s}[\boldsymbol{u}, \lambda] = \boldsymbol{0}. \tag{4.4}$$

Using previous notations, the shakedown problem in Definition 3.1 can be rewritten in discrete (approximate) form. We have to determine

$$\lambda_a := \max \lambda : \exists \boldsymbol{u} : \boldsymbol{s}[\boldsymbol{u}, \lambda] = \boldsymbol{0}. \tag{4.5}$$

So formulated, the shakedown problem looks very similar to the discrete formulation of the static theorem of limit analysis, the only difference being the role of the safety multiplier λ which acts as an internal parameter of function $f_s[\boldsymbol{\sigma}, \lambda]$ in spite of being an external load multiplier. However, this difference can hardly be considered meaningful. For instance, in the case of monotonous loading, we can write the limit analysis problem exactly in the form (4.5) if using the safety factor as multiplier for the elastic stress solution in place of a load multiplier.

The analogies between the two problems suggest that the methods available for solving limit analysis problems can be directly extended to the solution of shakedown ones. The solution method proposed in this paper actually follows this line. It can be considered a direct adaptation to shakedown of the so-called *strain driven* algorithm for incremental elastic–plastic analysis [39] and corresponds to a direct extension of the arc-length path-following method described in [14].

Some comments are useful here, before entering into the details of the proposed method. First of all, note that, constitutive laws being locally defined, the return mapping scheme

$$\boldsymbol{\sigma} := \boldsymbol{\sigma}_a[\boldsymbol{\sigma}_0 + \boldsymbol{E}\boldsymbol{\varepsilon}, \lambda], \quad \boldsymbol{\varepsilon} := \boldsymbol{D}(\boldsymbol{u} - \boldsymbol{u}_0) \tag{4.6}$$

can apply separately for each element (or Gauss point, if the element is defined by numerical integration). Therefore, the evaluation of the internal force vector $\boldsymbol{s}[\boldsymbol{u}, \lambda]$ for given $\boldsymbol{u}$ and λ through Eq. (4.1) is actually a very simple and computationally fast process. Some caution however has to be used in order to reduce discretization errors and avoid generating incoherencies. It is known in fact (see [7,8]) that strictly compatible formulations can produce discretization locking; equally, a careless use of collocation procedures can lose energy conservation laws.

A complete discussion of this topic is not possible here. As a general rule, we can suggest that discrete stress field $\boldsymbol{\sigma}$ be defined by extending Haar–Kármán equation (3.11) to the whole element, that is by the condition

$$\phi_e := \frac{1}{2}\int_{B_e} \{(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*)^{\mathrm{T}}\boldsymbol{E}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*)\}\,\mathrm{d}v = \min, \quad \boldsymbol{\sigma}[x] \in \mathbb{E}_s[\lambda], \qquad \forall x \in \{x_1, \ldots, x_n\}, \tag{4.7}$$

where B_e is the element volume, $\boldsymbol{\sigma}^* := \boldsymbol{\sigma}_0 + \boldsymbol{E}\boldsymbol{\varepsilon}$. $x_i, i = 1, \ldots, n$ is a discrete set of control points, such that to discretize, at least approximately, admissibility condition $\boldsymbol{\sigma} \in \mathbb{E}_s[\lambda], \forall x \in B_e$, and $\boldsymbol{\sigma}$ is self-equilibrated within the element or, at least, satisfies Eq. (2.1) in B_e for any $\delta\boldsymbol{\varepsilon} := \boldsymbol{D}\,\delta\boldsymbol{u}$. The implementation of condition (4.7) is quite easy. An example will be given in Section 5.3, with reference to the simple case of beam elements.

Note also that, as a consequence of Lemmas 3.1 and 3.4, two different states $(\boldsymbol{u}_1, \lambda_1)$ and $(\boldsymbol{u}_2, \lambda_2)$, satisfy the equation

$$\boldsymbol{\sigma}[\boldsymbol{u}_2, \lambda_2] - \boldsymbol{\sigma}[\boldsymbol{u}_1, \lambda_1] = \boldsymbol{E}_s[\lambda_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2](\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1) + \boldsymbol{\sigma}_s[\boldsymbol{\varepsilon}_1, \lambda_1, \lambda_2](\lambda_2 - \lambda_1),$$

where $\boldsymbol{\varepsilon}_1 = \boldsymbol{D}\boldsymbol{u}_1$, $\boldsymbol{\varepsilon}_2 = \boldsymbol{D}\boldsymbol{u}_2$. Then, by integrating on the volume and introducing the secant operators

$$\boldsymbol{K}_s[\lambda_2, \boldsymbol{u}_1, \boldsymbol{u}_2] := \int_B \boldsymbol{D}^{\mathrm{T}}\boldsymbol{E}_s[\lambda_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2]\boldsymbol{D}\,\mathrm{d}v, \qquad \boldsymbol{y}_s[\boldsymbol{u}_1, \lambda_1, \lambda_2] := \int_B \boldsymbol{D}^{\mathrm{T}}\boldsymbol{\sigma}_s[\boldsymbol{\varepsilon}_1, \lambda_1, \lambda_2]\,\mathrm{d}v, \tag{4.8}$$

we obtain

$$\boldsymbol{s}[\boldsymbol{u}_2, \lambda_2] - \boldsymbol{s}[\boldsymbol{u}_1, \lambda_1] = \boldsymbol{K}_s[\lambda_2, \boldsymbol{u}_1, \boldsymbol{u}_2](\boldsymbol{u}_2 - \boldsymbol{u}_1) + (\lambda_2 - \lambda_1)\boldsymbol{y}_s[\boldsymbol{u}_1, \lambda_1, \lambda_2]. \tag{4.9}$$

Due to Lemma 3.4, matrix $\boldsymbol{K}_s$ is obviously symmetric by construction and satisfies the conditions

$$0 \leqslant \delta\boldsymbol{u}^{\mathrm{T}}\boldsymbol{K}_s\,\delta\boldsymbol{u} \leqslant \delta\boldsymbol{u}^{\mathrm{T}}\boldsymbol{K}_e\,\delta\boldsymbol{u}, \quad \forall\delta\boldsymbol{u}. \tag{4.10}$$

Finally, it is worth mentioning that, the return mapping scheme (4.6) requires $\lambda_j \leqslant \bar{\lambda}, \forall j$. We can compute $\bar{\lambda}$ very easily as the smallest λ value that renders $\mathbb{E}_s$ void for some $x \in B$, which is obtained by scanning all control points, once for all, at the beginning of the analysis.

4.2. The proposed solution method

We determine the shakedown limit state, that is the limit shakedown multiplier λ_a, the related admissible self-equilibrated stress field $\boldsymbol{\sigma}_a$ and the corresponding induced displacement field $\boldsymbol{u}_a$, though a sequence of admissible safe states $\boldsymbol{x}^{(k)} := \{\lambda^{(k)}, \boldsymbol{\sigma}^{(k)}, \boldsymbol{u}^{(k)}\}$, $k = 1, 2, \ldots$, starting from the elastic limit state $\boldsymbol{x}^{(0)} := \{\lambda_e, \mathbf{0}, \mathbf{0}\}$, such that $\boldsymbol{s}^{(k)} := \boldsymbol{s}[\boldsymbol{u}^{(k)}, \lambda^{(k)}] = 0$, $\forall k$ and monotonous non decreasing in $\lambda^{(k)}$. The sequence is arrested when the limit shakedown state $\boldsymbol{x}_a$ is reached, which is obtained when $\lambda^{(k+1)} = \lambda^{(k)}$.

In each step of the process, the new state $\boldsymbol{x}^{(k)}$ is obtained from the previous one $\boldsymbol{x}^{(k-1)}$ using an iterative scheme which, starting from $\boldsymbol{x}_0 := \boldsymbol{x}^{(k-1)}$, produces a convergent sequence of values $\boldsymbol{x}_j := \{\lambda_j, \boldsymbol{\sigma}_j, \boldsymbol{u}_j\}$, $j = 1, 2, \ldots$ by recursively updating the displacement vector and the load multiplier:

$$\begin{cases} \boldsymbol{u}_{j+1} := \boldsymbol{u}_j + \dot{\boldsymbol{u}}_j, \\ \lambda_{j+1} := \lambda_j + \dot{\lambda}_j, \end{cases} \tag{4.11}$$

corrections $\dot{\boldsymbol{u}}_j$ and $\dot{\lambda}_j$ being defined in order to satisfy, at least approximately, the equilibrium condition $\boldsymbol{s}[\boldsymbol{u}_{j+1}, \lambda_{j+1}] = 0$ required by Eq. (4.5), while $\boldsymbol{\sigma}_{j+1}$ is defined by the return mapping (4.6).

We have already seen that obtaining $\boldsymbol{\sigma}_{j+1}$ from λ_{j+1} and $\boldsymbol{u}_{j+1}$ is quite straightforward, so we can concentrate on the updating of $\boldsymbol{u}_j$ and λ_j, which is obtained by defining corrections $\dot{\boldsymbol{u}}_j$ and $\dot{\lambda}_j$ as the solution of the linear system

$$\begin{bmatrix} \boldsymbol{K}_e & \boldsymbol{y}_j \\ \boldsymbol{y}_j^{\mathrm{T}} & 0 \end{bmatrix}, \quad \begin{bmatrix} \dot{\boldsymbol{u}}_j \\ \dot{\lambda}_j \end{bmatrix} = \begin{bmatrix} -\boldsymbol{s}_j \\ 0 \end{bmatrix}, \quad \boldsymbol{s}_j := \boldsymbol{s}[\boldsymbol{u}_j, \lambda_j], \tag{4.12}$$

where $\boldsymbol{K}_e$ is the elastic stiffness matrix defined by Eq. (4.3) and vector $\boldsymbol{y}_j$ is defined, according to Eq. (4.8), as

$$\boldsymbol{y}_j := \int_B \mathbf{D}^{\mathrm{T}} \boldsymbol{\sigma}_s[\boldsymbol{\varepsilon}_j, \lambda_j, \lambda_{j+1}] \, \mathrm{d}v = \frac{1}{\lambda_{j+1} - \lambda_j} (\boldsymbol{s}[\boldsymbol{u}_j, \lambda_{j+1}] - \boldsymbol{s}[\boldsymbol{u}_j, \lambda_j]). \tag{4.13}$$

Eq. (4.12) is conveniently solved by partitioning. In explicit form, we obtain:

$$\dot{\lambda}_j = -\frac{\boldsymbol{y}_j^{\mathrm{T}} \boldsymbol{K}_e^{-1} \boldsymbol{s}_j}{\boldsymbol{y}_j^{\mathrm{T}} \boldsymbol{K}_e^{-1} \boldsymbol{y}_j}, \qquad \dot{\boldsymbol{u}}_j = -\boldsymbol{K}_e^{-1} \boldsymbol{s}_j - \dot{\lambda}_j \boldsymbol{K}_e^{-1} \boldsymbol{y}_j. \tag{4.14}$$

Note that $\lambda_{j+1} = \lambda_j + \dot{\lambda}_j$ is required in Eq. (4.13) for obtaining $\boldsymbol{y}_j$, so Eq. (4.13) couples with the first of (4.14). However, λ_{j+1} can easily be obtained by iteration as the limit of the sequence

$$\tilde{\lambda}_i := \lambda_j - \frac{\tilde{\boldsymbol{y}}_i^{\mathrm{T}} \boldsymbol{K}_e^{-1} \boldsymbol{s}_j}{\tilde{\boldsymbol{y}}_i^{\mathrm{T}} \boldsymbol{K}_e^{-1} \tilde{\boldsymbol{y}}_i}, \qquad \tilde{\boldsymbol{y}}_i := \frac{1}{\tilde{\lambda}_i - \lambda_j} (\boldsymbol{s}[\boldsymbol{u}_j, \tilde{\lambda}_i] - \boldsymbol{s}_j) \tag{4.15}$$

which is initialized by assuming the first evaluation for $\boldsymbol{y}_j$ be defined as the initial tangent

$$\boldsymbol{y}_j \approx \tilde{\boldsymbol{y}}_1 := \boldsymbol{y}[\boldsymbol{u}_j, \lambda_j, \lambda_j]. \tag{4.16}$$

The sequence implements the secant iteration, so it is always convergent. Actually it converges very fast and, considering that its solution is used within an external iteration scheme, only one loop is usually needed. For obvious reasons, it is also convenient to introduce the constraint $\lambda_j + \dot{\lambda}_j \leqslant \bar{\lambda}$.

The proposed solution algorithm is described in pseudo-code form in Table 1. Note that the iterative process (4.11) and (4.12), for the kth step, is initialized by assuming

$$\begin{aligned} \lambda_1 &:= \lambda^{(k-1)} + \beta^{(k)} (\lambda^{(k-1)} - \lambda^{(k-2)}), \\ \boldsymbol{u}_1 &:= \boldsymbol{u}^{(k-1)} + \beta^{(k)} (\boldsymbol{u}^{(k-1)} - \boldsymbol{u}^{(k-2)}), \end{aligned} \tag{4.17}$$

$\beta^{(k)}$ being an appropriate scaling factor we can relate to iteration loops performed in the previous step by taking

Table 1
Pseudo-code for the proposed solution algorithm

1. Assemble and decompose matrix $\boldsymbol{K}_E$
2. Compute the elastic solutions $(\boldsymbol{u}_i^e, \boldsymbol{\sigma}_i^e)$ for each basic load $\boldsymbol{p}_i$

$$\boldsymbol{u}_i := \boldsymbol{K}_E^{-1}\mathbf{p}_i, \qquad \boldsymbol{\sigma}_i := \boldsymbol{E}\mathbf{D}\boldsymbol{u}_i.$$

3. Compute the elastic limit multiplier λ_e and initialize the solution process

$$\boldsymbol{u}^{(0)} := \mathbf{0},\ \boldsymbol{\sigma}^{(0)} = \mathbf{0},\ \lambda^{(0)} := \mathbf{0},\ \boldsymbol{u}^{(1)} := \mathbf{0},\ \boldsymbol{\sigma}^{(1)} = \mathbf{0},\ \lambda^{(1)} := \lambda_e.$$

4. *Repeat for* $k = 2, 3, \ldots$
   (a) Evaluate, using Eq. (4.17):
   $$\lambda_1 := \lambda^{(k-1)} + \beta(\lambda^{(k-1)} - \lambda^{(k-2)}),$$
   $$\boldsymbol{u}_1 := \boldsymbol{u}^{(k-1)} + \beta(\boldsymbol{u}^{(k-1)} - \boldsymbol{u}^{(k-2)}).$$
   (b) *Repeat for* $j = 1, 2, \ldots$
   - Compute $\boldsymbol{\sigma}_j$ through Eq. (4.6)
   - Compute vectors $\boldsymbol{s}_j$ and $\boldsymbol{y}_j$ through Eqs. (4.12) and (4.16)
   - Evaluate the new estimate $\boldsymbol{x}_{j+1}$:
   $$\begin{cases} \lambda_{j+1} = \lambda_j + \dot{\lambda}_j \\ \boldsymbol{u}_{j+1} = \boldsymbol{u}_j - \mathbf{K}_E^{-1}(\boldsymbol{s}_j + \dot{\lambda}_j \boldsymbol{y}_j) \end{cases}, \qquad \dot{\lambda}_j = -\frac{\boldsymbol{y}_j^{\mathrm{T}} \boldsymbol{K}_E^{-1} \boldsymbol{s}_j}{\boldsymbol{y}_j^{\mathrm{T}} \boldsymbol{K}_E^{-1} \boldsymbol{y}_j}.$$
   Until $\|\boldsymbol{s}_j\| \leqslant \mathrm{tol}_1$
   (c) Update the solution
   $$\lambda^{(k)} = \lambda_j, \quad \boldsymbol{u}^{(k)} := \boldsymbol{u}_j, \quad \boldsymbol{\sigma}^{(k)} := \boldsymbol{\sigma}_j.$$
   Until $\lambda^{(k)} - \lambda^{(k-1)} \leqslant \mathrm{tol}_2$
5. Obtain the solution of the shakedown problem

$$\lambda_a := \lambda^{(k)}, \quad \boldsymbol{u}_a := \boldsymbol{u}^{(k)}, \quad \boldsymbol{\sigma}_a := \boldsymbol{\sigma}^{(k)}.$$

$$\beta^{(k)} = \frac{n^{(k-1)} - \bar{n}}{n^{(k-1)} + \bar{n}},$$

where $n^{(k-1)}$ is the number of loops performed in the previous step and $\bar{n}$ its assumed reference value (typically $\bar{n} \approx 4, \ldots, 8$). The same formula is used in the first step of the algorithm by making

$$\{\boldsymbol{u}^{(0)} = 0, \boldsymbol{\sigma}^{(0)} = 0, \lambda^{(0)} = 0\}, \qquad \{\boldsymbol{u}^{(1)} = 0, \boldsymbol{\sigma}^{(1)} = 0, \lambda^{(1)} = \lambda_e\}$$

and assuming β_1 as a small fraction ($\beta_1 \approx 0.01$).

The iteration process stops and the step is assumed to have been performed when

$$\|\boldsymbol{s}_j\| \leqslant \mathrm{tol}_1, \qquad \|\boldsymbol{s}\|^2 := \boldsymbol{s}^{\mathrm{T}} \boldsymbol{K}_e^{-1} \boldsymbol{s}. \tag{4.18}$$

Finally, the algorithm is arrested when $\lambda^{(k)} = \lambda^{(k-1)}$, within the assigned tolerance tol_2. As we will see, that implies the required shakedown solution has already been reached.

In the following subsections we will show the effectiveness of this strategy by proving that:

1. The sequence $\boldsymbol{x}_j$ produced by the iterative scheme (4.11) and (4.12) converges to a new solution $\boldsymbol{x}^{(k)} := \{\lambda^{(k)}, \boldsymbol{\sigma}^{(k)}, \boldsymbol{u}^{(k)}\}$, characterized by $\boldsymbol{\sigma}^{(k)} \in \mathbb{E}_s[\lambda^{(k)}]$ and $\boldsymbol{s}[\boldsymbol{u}^{(k)}, \lambda^{(k)}] = 0$. Then, according to Eq. (4.5), it provides a lower bound λ_s for the shakedown safety factor λ_a.
2. The succession of shakedown states $\boldsymbol{x}^{(k)}$ satisfies the condition $\lambda^{(k)} \geqslant \lambda^{(k-1)}$. Furthermore, the occurrence of $\lambda^{(k)} = \lambda^{(k-1)}$ implies $\lambda^{(k)} = \lambda_a$, so providing the shakedown solution.

4.3. Convergence of the iterative scheme (4.11) and (4.12)

In order to discuss the convergence of the iterative scheme, it is convenient to refer to the solution $\{\mu_i, \boldsymbol{z}_i\}$ of the eigenvalue problem

$$[\mu \boldsymbol{K}_e - \boldsymbol{K}_j]\boldsymbol{z} = 0.$$

Both matrices $\boldsymbol{K}_j$ and $\boldsymbol{K}_e$ being symmetric and the first being positive defined, we know that μ_i and $\boldsymbol{z}_i$ are characterized by the conditions:

$$\boldsymbol{z}_i^{\mathrm{T}}\boldsymbol{K}_e\boldsymbol{z}_j = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \neq j, \end{cases} \qquad \boldsymbol{z}_i^{\mathrm{T}}\boldsymbol{K}_j\boldsymbol{z}_j = \begin{cases} \mu_i & \text{if } i = 0, \\ 0 & \text{if } i \neq j. \end{cases} \tag{4.19}$$

Obviously, due to Eq. (4.10) we also have

$$0 \leqslant \mu_i \leqslant 1, \quad \forall i. \tag{4.20}$$

We are now ready to prove the following theorem:

Theorem 4.1 (convergence of the iterative scheme to a self-equilibrated solution). *The sequence $\boldsymbol{x}_j$ produced by the iterative scheme* (4.11) *and* (4.12) *is convergent and its limit $\boldsymbol{x}^{(k)} := \{\lambda^{(k)}, \boldsymbol{\sigma}^{(k)}, \boldsymbol{u}^{(k)}\}$ is characterized by $\boldsymbol{\sigma}^{(k)} \in \mathbb{E}_s[\lambda^{(k)}]$ and $\boldsymbol{s}[\boldsymbol{u}^{(k)}, \lambda^{(k)}] = 0$.*

Proof. By definition (4.14) we have

$$\boldsymbol{s}_{j+1} - \boldsymbol{s}_j = \boldsymbol{K}_j\dot{\boldsymbol{u}}_j + \dot{\lambda}_j\boldsymbol{y}_j, \quad \dot{\lambda}_j := -\frac{\boldsymbol{y}_j^{\mathrm{T}}\boldsymbol{K}_e^{-1}\boldsymbol{s}_j}{\boldsymbol{y}_j^{\mathrm{T}}\boldsymbol{K}_e^{-1}\boldsymbol{y}_j}, \quad \dot{\boldsymbol{u}}_j := -\boldsymbol{K}_e^{-1}(\boldsymbol{s}_j + \dot{\lambda}_j\boldsymbol{y}_j)$$

that provides

$$\boldsymbol{s}_{j+1} = \boldsymbol{s}_j + \dot{\lambda}_j\boldsymbol{y}_j - \boldsymbol{K}_j\boldsymbol{K}_e^{-1}(\boldsymbol{s}_j + \dot{\lambda}_j\boldsymbol{y}_j).$$

Then, expanding $\boldsymbol{y}_j$, $\boldsymbol{s}_j$ and $\boldsymbol{s}_{j+1}$ in terms of $\boldsymbol{z}_i$

$$\boldsymbol{y}_j = \boldsymbol{K}_e\sum_i \beta_i\boldsymbol{z}_i, \quad \boldsymbol{s}_j = \boldsymbol{K}_e\sum_i \alpha_i\boldsymbol{z}_i, \quad \boldsymbol{s}_{j+1} = \boldsymbol{K}_e\sum_i \tilde{\alpha}_i\boldsymbol{z}_i, \tag{4.21}$$

we obtain, due to Eq. (4.19),

$$\tilde{\alpha}_i = (1-\mu_i)(\alpha_i + \dot{\lambda}_j\beta_i) \tag{4.22}$$

and therefore

$$\|\boldsymbol{s}_{j+1}\|^2 = \sum_i \tilde{\alpha}_i^2 = \sum_i (1-\mu_i)^2(\alpha_i^2 + \dot{\lambda}_j^2\beta_i^2 + 2\dot{\lambda}_j\alpha_i\beta_i). \tag{4.23}$$

Due to Eq. (4.1), we have

$$\boldsymbol{y}_j^{\mathrm{T}}\dot{\boldsymbol{u}}_j = -\boldsymbol{y}_j^{\mathrm{T}}\boldsymbol{K}_e^{-1}(\boldsymbol{s}_j + \dot{\lambda}\boldsymbol{y}_j) = 0 \Rightarrow \sum_i \beta_i(\alpha_i + \dot{\lambda}_j\beta_i) = 0,$$

so we can simplify Eq. (4.23) into

$$\|\boldsymbol{s}_{j+1}\|^2 = \sum_i (1-\mu_i)^2(\alpha_i^2 - \dot{\lambda}_j^2\beta_i^2).$$

By comparison with

$$\|\boldsymbol{s}_j\|^2 = \sum_i \alpha_i^2$$

and using Eq. (4.20), we obtain

$$\|\boldsymbol{s}_{j+1}\| \leqslant \|\boldsymbol{s}_j\|$$

while

$$\frac{\|\boldsymbol{s}_{j+1}\|}{\|\boldsymbol{s}_j\|} < 1 \quad \text{if } \exists i : \mu_i \alpha_i \neq 0 \text{ or } \dot{\lambda}_j \neq 0.$$

We then stated that the sequence $\{\|\boldsymbol{s}_j\|\}$ is monotonously non increasing. It is also bounded from below ($\|\boldsymbol{s}_j\| \geqslant 0$) so it is convergent, while not necessarily to the limit $\|\boldsymbol{s}\| = 0$. We can prove, however, that the occurrence $\lim_j \|\boldsymbol{s}_j\| > 0$ is impossible. In fact, assuming

$$\lim_{j\to\infty} \|\boldsymbol{s}_j\| > 0,$$

this implies

$$\lim_{j\to\infty} \dot{\lambda}_j = 0, \qquad \lim_{j\to\infty} \mu_i \alpha_i = 0, \quad \forall i$$

and, because of Eqs. (4.21) and (4.22),

$$\lim_{j\to\infty} (\boldsymbol{s}_{j+1} - \boldsymbol{s}_j) = 0.$$

The sequence $\{\boldsymbol{s}_j\}$ is then convergent and, by assumption,

$$\boldsymbol{s}^{(k)} = \lim_{j\to\infty} \boldsymbol{s}_j \neq \mathbf{0}.$$

We obtain, because of Eqs. (4.14) and (4.11)

$$\lim_{j\to\infty} (\boldsymbol{u}_{j+1} - \boldsymbol{u}_j) = \bar{\boldsymbol{u}} := \boldsymbol{K}_e^{-1} \boldsymbol{s}^{(k)} \neq \mathbf{0}$$

while having

$$\lim_{j\to\infty} \int_B (\boldsymbol{\sigma}_{j+1} - \boldsymbol{\sigma}_j)^{\mathrm{T}} (\boldsymbol{\varepsilon}_{j+1} - \boldsymbol{\varepsilon}_j)\, \mathrm{d}v = \lim_{j\to\infty} (\boldsymbol{s}_{j+1} - \boldsymbol{s}_j)^{\mathrm{T}} (\boldsymbol{u}_{j+1} - \boldsymbol{u}_j) = 0.$$

Due to Lemma 3.2, the latter implies $\lim_{j\to\infty} (\boldsymbol{\sigma}_{j+1} - \boldsymbol{\sigma}_j) = 0$, therefore

$$\lim_{j\to\infty} (\boldsymbol{\varepsilon}^{\mathrm{p}}_{j+1} - \boldsymbol{\varepsilon}^{\mathrm{p}}_j) := \lim_{j\to\infty} (\boldsymbol{\varepsilon}_{j+1} - \boldsymbol{\varepsilon}_j) = \bar{\boldsymbol{\varepsilon}} := \boldsymbol{D}\bar{\boldsymbol{u}} \neq \mathbf{0}.$$

As a consequence, we should have

$$\lim_{j\to\infty} \boldsymbol{\varepsilon}^{\mathrm{p}}_{j+k} = \boldsymbol{\varepsilon}^{\mathrm{p}}_j + k\bar{\boldsymbol{\varepsilon}} \Rightarrow \lim_{j,k\to\infty} \frac{1}{k} \boldsymbol{\varepsilon}^{\mathrm{p}}_{j+k} = \bar{\boldsymbol{\varepsilon}},$$

but this implies

$$\lim_{j,k\to\infty} (\boldsymbol{u}_{j+k+1} - \boldsymbol{u}_{j+k})^{\mathrm{T}} \boldsymbol{y}_{j+k} = \bar{\boldsymbol{u}}^{\mathrm{T}} \boldsymbol{y}_{j+k} = \frac{1}{k} \int_B \boldsymbol{\varepsilon}^{\mathrm{p}}_{j+k} \boldsymbol{\sigma}_s[\boldsymbol{\varepsilon}_{j+k}, \lambda_{j+k}, \lambda_{j+k}]\, \mathrm{d}v < 0$$

which is absurd being in contrast with the Eqs. (4.11) and (4.12) which give $(\boldsymbol{u}_{j+k+1} - \boldsymbol{u}_{j+k})^{\mathrm{T}} \boldsymbol{y}_{j+k} = 0, \forall j, k$.

Therefore, we necessarily have $\lim_{j\to\infty} \|\boldsymbol{s}_j\| = 0$. This implies the convergence of $\{\boldsymbol{s}_j\}$ to a null vector

$$\boldsymbol{s}^{(k)} := \lim_{j\to\infty} \boldsymbol{s}_j = \mathbf{0}$$

and, as a consequence,

$$\lim_{j\to\infty}(\boldsymbol{u}_{j+1}-\boldsymbol{u}_j)=\lim_{j\to\infty}\boldsymbol{K}_e^{-1}\boldsymbol{s}_j=\boldsymbol{0},\qquad \lim_{j\to\infty}(\lambda_{j+1}-\lambda_j)=\lim_{j\to\infty}\frac{\boldsymbol{y}_j^{\mathrm{T}}\boldsymbol{s}_j}{\boldsymbol{y}_j^{\mathrm{T}}\boldsymbol{y}_j}=0.$$

Both sequences $\{\boldsymbol{u}_j\}$ and $\{\lambda_j\}$ are then also convergent and we can define

$$\boldsymbol{u}^{(k)}=\lim_{j\to\infty}\boldsymbol{u}_j,\qquad \lambda^{(k)}=\lim_{j\to\infty}\lambda_j.$$

Furthermore, $\boldsymbol{\sigma}_a[\boldsymbol{u},\lambda]$ being continuous in both $\boldsymbol{u}$ and λ, $\boldsymbol{\sigma}_j:=\boldsymbol{\sigma}_a[\boldsymbol{u}_j,\lambda_j]$ is also convergent. So, being $\boldsymbol{\sigma}_j\in\mathbb{E}_s[\lambda_j]$, $\forall j$ by definition, we have

$$\boldsymbol{\sigma}^{(k)}=\lim_{j\to\infty}\boldsymbol{\sigma}_j\in\mathbb{E}_s[\lambda^{(k)}]$$

which completes the proof. □

4.4. Convergence to the shakedown solution

The convergence of the sequence $\boldsymbol{x}^{(k)}$ produced by the incremental process to the required shakedown solution is stated by the following theorem:

Theorem 4.2 (convergence of the incremental process to the shakedown solution). *The sequence $\lambda^{(k)}$ is characterized by*

$$\lambda^{(k-1)}\leqslant\lambda^{(k)}\leqslant\lambda_a.$$

Furthermore, the occurrence of $\lambda^{(k)}=\lambda^{(k-1)}$ and $\boldsymbol{u}^{(k)}\neq\boldsymbol{u}^{(k-1)}$ implies the achievement of the shakedown solution:

$$\lambda_a=\lambda^{(k)}.$$

Proof. We have, $\boldsymbol{\sigma}^{(k)}\in\overline{\mathbb{S}}$ (according to Eq. (4.4)) and $\boldsymbol{\sigma}^{(k)}\in\mathbb{E}_s$ for any k, by definition. So condition $\lambda^{(k)}\leqslant\lambda_a$ comes directly from Eq. (3.7). We also obtain

$$\int_B(\boldsymbol{\sigma}^{(k)}-\boldsymbol{\sigma}^{(k-1)})^{\mathrm{T}}\boldsymbol{\varepsilon}^{(k)}\,\mathrm{d}v=(\boldsymbol{s}^{(k)}-\boldsymbol{s}^{(k-1)})^{\mathrm{T}}(\boldsymbol{u}^{(k)}-\boldsymbol{u}^{(k-1)})=0.$$

Then, by letting

$$\boldsymbol{\varepsilon}^{(k)}=\boldsymbol{\varepsilon}^{(ke)}+\boldsymbol{\varepsilon}^{(kp)}$$

and remembering Eq. (4.1), we have

$$\int_B(\boldsymbol{\sigma}^{(k)}-\boldsymbol{\sigma}^{(k-1)})^{\mathrm{T}}\boldsymbol{\varepsilon}^{(kp)}\,\mathrm{d}v=-\int_B\boldsymbol{\varepsilon}^{(ke)^{\mathrm{T}}}\boldsymbol{E}\boldsymbol{\varepsilon}^{(ke)}\,\mathrm{d}v\leqslant 0.$$

This implies that $\lambda^{(k)}\geqslant\lambda^{(k-1)}$. In fact, if assuming $\lambda^{(k)}<\lambda^{(k-1)}$, $\boldsymbol{\sigma}^{(k-1)}\in\mathbb{E}_s[\lambda^{(k-1)}]$ is internal to $\mathbb{E}_s[\lambda^{(k)}]$ because of Eq. (3.3), then we should have, due to Drucker's inequality (3.4),

$$\int_B(\boldsymbol{\sigma}^{(k)}-\boldsymbol{\sigma}^{(k-1)})^{\mathrm{T}}\boldsymbol{\varepsilon}^{(kp)}\,\mathrm{d}v>0$$

which is a contradiction.

The case $\lambda^{(k)}=\lambda^{(k-1)}$ is only possible if $\boldsymbol{\varepsilon}^{(ke)}=0$. This implies

$$\boldsymbol{\varepsilon}^{(kp)}=\boldsymbol{\varepsilon}^{(k)}=\boldsymbol{D}(\boldsymbol{u}^{(k)}-\boldsymbol{u}^{(k-1)})$$

that is, $\boldsymbol{\varepsilon}^{(kp)} \in \mathbb{K}$ and $\boldsymbol{\varepsilon}^{(kp)} \neq 0$. Remembering that $\boldsymbol{\sigma}^{(k)} \in \overline{\mathbb{S}}$, we have

$$\int_B \boldsymbol{\sigma}^{(k)^{\mathrm{T}}} \boldsymbol{\varepsilon}^{(kp)} \,\mathrm{d}v = 0,$$

$\boldsymbol{\sigma}^{(k)}$ and $\boldsymbol{\varepsilon}^{(kp)}$ being associated by flow rule (3.10), by construction. The condition implies $\lambda^{(k)} \geqslant \lambda_a$, because of Eq. (3.8). Conversely, we have $\lambda^{(k)} \leqslant \lambda_a$, as shown before. Therefore, we obtain $\lambda^{(k)} = \lambda_a$, as stated by the theorem. □

It is worth mentioning that the step-length is naturally forced by the arc-length strategy (4.17), (4.11) and (4.12), so the solution process does not need any special artifice for avoiding a premature stop due to the occurrence of $\boldsymbol{u}^{(k)} = \boldsymbol{u}^{(k-1)}$.

4.5. A comment about computational updating of domain $\mathbb{E}_s[\lambda]$

The shakedown admissible domain $\mathbb{E}_s[\lambda]$ defined in Section 3.2 is a key ingredient of the proposed method. It has to be updated according to λ_j at each iteration loop of the incremental process. The computational efficiency of its updating plays an important role in the analysis and specialized, fully optimized problem-dependent formulas should be used in general.

We discuss here the case when the yield conditions are piecewise linearized, that is, function $f[\boldsymbol{\sigma}]$ reduces to a the m-faces polyhedron:

$$f[\boldsymbol{\sigma}] := \max_k \left\{ \boldsymbol{n}_k^{\mathrm{T}} \boldsymbol{\sigma} - c_k \right\} \leqslant 0, \quad k = 1, \ldots, m. \tag{4.24}$$

A very simple and fast formula is obtained in this case. According to Eq. (3.1), we have

$$f_s[\boldsymbol{\sigma}, \lambda] = \max_k \left\{ \boldsymbol{n}_k^{\mathrm{T}} \boldsymbol{\sigma} - c_k + b_k[\lambda] \right\}, \quad b_k[\lambda] := \max_{\sigma_e \in \mathbb{S}_e} \left\{ \boldsymbol{n}_k^{\mathrm{T}} \boldsymbol{\sigma}_e \right\}.$$

The elastic stress envelope $\mathbb{S}_e$ being defined by the polyhedron

$$\mathbb{S}_e := \left\{ \boldsymbol{\sigma}_e[t] := \sum_{i=1}^{p} \alpha_i[t] \boldsymbol{\sigma}_{ei} : \alpha_i^{\min} \leqslant \alpha_i[t] \leqslant \alpha_i^{\max}, \ \forall t \right\},$$

we have

$$b_k[\lambda] = \lambda \sum_{i=1}^{p} a_{ki}, \qquad a_{ki} := \left\langle \begin{array}{ll} \alpha_i^{\min} \boldsymbol{n}_k^{\mathrm{T}} \boldsymbol{\sigma}_{ei} & \text{if } \boldsymbol{n}_k^{\mathrm{T}} \boldsymbol{\sigma}_{ei} < 0, \\ \alpha_i^{\max} \boldsymbol{n}_k^{\mathrm{T}} \boldsymbol{\sigma}_{ei} & \text{if } \boldsymbol{n}_k^{\mathrm{T}} \boldsymbol{\sigma}_{ei} \geqslant 0, \end{array} \right.$$

so $f_s[\boldsymbol{\sigma}, \lambda]$ is simply obtained in the form

$$f_s[\boldsymbol{\sigma}, \lambda] = \max_k \left\{ \boldsymbol{n}_k^{\mathrm{T}} \boldsymbol{\sigma} - c_k + \lambda \bar{b}_k \right\}, \tag{4.25}$$

where $\bar{b}_k := \sum_{i=1}^{p} a_{ki}$ is computed, once and for all, at the beginning of the process.

4.6. Some further comments

The described solution process can be viewed as a step-by-step incremental process aiming at simulating the case of a proportional increasing loading evolution such that, in each step k, the load recycles within all possible values in $\lambda^{(k)}\mathbb{P}$ up to the achievement of elastic adaptation.

Within this interpretation, the proposed solution process corresponds to a standard path-following incremental process. Iteration scheme (4.11) and (4.12) actually corresponds to an implementation of the Riks arc-length scheme [16,17]. We know (e.g. see [40]) that this scheme can be written in the form

$$\begin{bmatrix} \widetilde{\boldsymbol{K}} & \tilde{\boldsymbol{y}} \\ \Delta\boldsymbol{u}^{\mathrm{T}}\boldsymbol{M} & \gamma \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{u}}_j \\ \dot{\lambda}_j \end{Bmatrix} = \begin{Bmatrix} -\boldsymbol{s}_j \\ 0 \end{Bmatrix},$$

where matrix $\widetilde{\boldsymbol{K}}$ and vector $\tilde{\boldsymbol{y}}$ are approximations for the Hessian $\boldsymbol{K}_t := \partial\boldsymbol{s}/\partial\boldsymbol{u}$ and $\boldsymbol{y}_t := \partial\boldsymbol{s}/\partial\lambda$, $\Delta\boldsymbol{u} := \boldsymbol{u}_j - \boldsymbol{u}_0$ and $\boldsymbol{M}$ and γ express the metric used for measuring the distance in the $\{\boldsymbol{u}, \lambda\}$ space:

$$\|\{\Delta\boldsymbol{u}, \Delta\lambda\}\|^2 := \begin{Bmatrix} \Delta\boldsymbol{u} \\ \Delta\lambda \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{M} & \cdot \\ \cdot & \gamma \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{u} \\ \Delta\lambda \end{Bmatrix}.$$

So Eq. (4.12) corresponds to the following assumptions:

$$\boldsymbol{y}_t := \boldsymbol{y}_j, \quad \widetilde{\boldsymbol{K}} := \boldsymbol{K}_e, \quad \boldsymbol{M}\Delta\boldsymbol{u} := \boldsymbol{y}_j, \quad \gamma = 0.$$

Different, even better, choices are obviously possible and could be usefully investigated. Actually, the convenience of our assumptions lies in the fact that, with the particular choice made, we can easily prove the convergence of both the iteration scheme and the solution process. It is worth mentioning that a similar assumption has been proposed in [14] for implementing a path-following solution process for limit analysis problems.

It is also useful to note that, in addition to $\lambda^{(k)}$ and $\boldsymbol{\sigma}^{(k)}$, the proposed solution process also produces a solution in terms of both displacements $\boldsymbol{u}^{(k)}$ and total (cumulated) plastic strains

$$\sum_{i=0}^{k} \boldsymbol{\varepsilon}^{(ip)} := \sum_{i=0}^{k} \left(\boldsymbol{D}\boldsymbol{u}^{(i)} - \boldsymbol{E}^{-1}\boldsymbol{\sigma}^{(i)}\right).$$

Obviously, these quantities are strictly related to the loading evolution law which is actually simulated and then only define lower bounds. However they can provide useful information about the elastic adaptation process and can be taken as reference values to be used, within a conventional safety criterion, for evaluating safety factors for displacement and plastic deformation limit states (the available upper bounds on plastic strains [42–44] appears to wide for being suitable for technical purposes). Relations with the Ponter–Martin extremal paths theory [41] could also be usefully investigated.

Finally, it is worth mentioning that shakedown analysis reduces to limit analysis when the load domain simplifies to a single point $\boldsymbol{p}$. So the proposed solution process can be directly used in limit analysis problems. Actually it can be viewed as a slight variation of a standard path-following process for proportionally increasing loads.

Differences are related to the use of the additional stress $\boldsymbol{\sigma} \in \mathbb{E}_s[\lambda]$. Path-following processes aiming to recover limit analysis usually refer to the total stress $\boldsymbol{\sigma} \in \mathbb{E}$ and use λ directly as a load amplifier (see [14,17]). So equilibrium is expressed by the condition

$$\boldsymbol{r}[\boldsymbol{u}, \lambda] := \boldsymbol{s}[\boldsymbol{u}] - \lambda\boldsymbol{p} = 0,$$

$\boldsymbol{s}$ being the nodal response due to the plastically admissible stress field $\boldsymbol{\sigma}$ associated to $\boldsymbol{u}$, expressed in our notations by

$$\boldsymbol{\sigma} := \boldsymbol{\sigma}_a[\boldsymbol{\sigma}_0 + \boldsymbol{E}\boldsymbol{\varepsilon}, 0].$$

As a consequence, the correction scheme (4.12) becomes (see [14])

$$\begin{bmatrix} \boldsymbol{K}_e & -\boldsymbol{p} \\ -\boldsymbol{p}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}}_j \\ \dot{\lambda}_j \end{bmatrix} = \begin{bmatrix} -\boldsymbol{r}[\boldsymbol{u}_j, \lambda_j] \\ 0 \end{bmatrix}, \tag{4.26}$$

where $-\boldsymbol{p} = \partial\boldsymbol{r}/\partial\lambda$ is used in place of $\boldsymbol{y}_j$. Obviously, for every kinematically compatible plastic strain field $\boldsymbol{\varepsilon}^{\mathrm{p}} := \boldsymbol{D}\boldsymbol{u}^{\mathrm{p}}$, we have

$$\int_B \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon}^{\mathrm{p}}\,\mathrm{d}v = \boldsymbol{p}^{\mathrm{T}}\boldsymbol{u}^{\mathrm{p}} > 0,$$

so this substitution also guarantees convergence for the same motivations as in Theorems 4.1 and 4.2. We can expect quite similar behaviour and convergence properties for the two schemes, as confirmed by the numerical tests shown in Section 6.4. Note however that scheme (4.12) requires the computation of vector $\boldsymbol{y}_j$ at each loop that is skipped by the standard scheme (4.26), so it is slightly more expensive. For this reason it has to be considered as not particularly convenient for limit analysis problems, that is when we are in the presence of a single loading case.

5. Shakedown analysis of plane frames

We now discuss some of the implementation details of the proposed solution method by referring to the shakedown analysis of plane frames. This is a very simple but also technically meaningful application context, so it is particularly suitable for exemplification purposes.

5.1. Finite element discretization

Let us consider a beam element and indicate with indices i and $j$ths end-sections. Assuming a local reference system $\{x, y\}$ with the x axes coincident with the beam axis, the kinematics of the beam is described by the motion of the normal sections in the $\{x, y\}$ plane, that is, by its axial $u[x]$, transversal $w[x]$ and rotational $\varphi[x]$ components. Stresses are described by the strength fields $N[x]$, $T[x]$ and $M[x]$, which express the axial and shear forces and the bending moment acting on beam sections, respectively. Obviously, a self-equilibrated stress field is expressed by

$$N[x] = \text{constant}, \quad M[x] = \text{linear}, \quad T[x] = M_{,x} = \text{constant}. \tag{5.1}$$

It is convenient to refer to the so-called *natural stresses*:

$$m_a := N\ell, \quad m_s := M_i + M_j, \quad m_e := M_i - M_j. \tag{5.2}$$

We have:

$$N[x] = \frac{m_a}{\ell}, \quad T[x] = -\frac{m_e}{\ell}, \quad M[x] = \frac{1}{2}(m_s + m_e(1 - 2\xi)), \tag{5.3}$$

$\ell := x_j - x_i$ being the element length and $\xi := (x - x_i)/\ell$ a non-dimensional abscissa varying in $[0 \cdots 1]$.

The element strain energy can be easily expressed as

$$\Phi_b := \frac{\ell}{2}\int_0^1 \left(\frac{N^2}{EA} + \frac{T^2}{GA^*} + \frac{M^2}{EJ}\right) \mathrm{d}\xi, \tag{5.4}$$

where A, A^* and J are the area, the equivalent shear area and the inertia of the section, and E and G the normal and transversal elastic moduli. We also have, from Clapeyron's theorem,

$$\Phi_b = \tfrac{1}{2}\big(N(u_j - u_i) + T(w_j - w_i) + M_i\varphi_i - M_j\varphi_j\big).$$

Therefore, by introducing the associated *natural strains*

$$\phi_a := (u_j - u_i)/\ell, \quad \phi_s := (\varphi_i - \varphi_j)/2, \quad \phi_e := (\varphi_i + \varphi_j)/2 - (w_j - w_i)/\ell, \tag{5.5}$$

we can write

$$\Phi_b = \frac{1}{2}\begin{Bmatrix} m_a \\ m_s \\ m_e \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \phi_a \\ \phi_s \\ \phi_e \end{Bmatrix}. \tag{5.6}$$

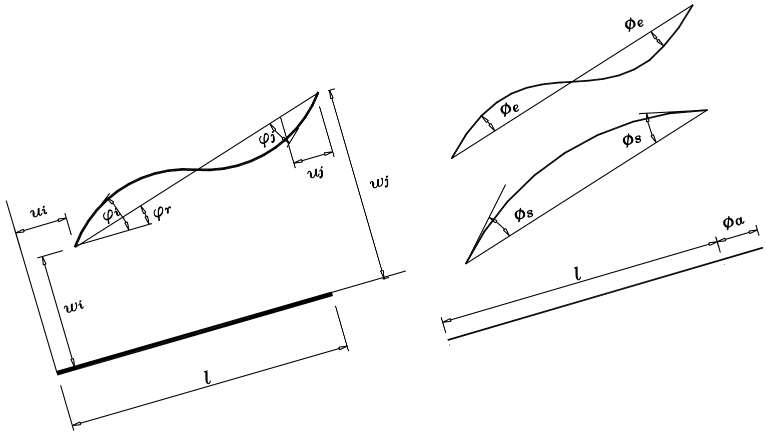

Fig. 3. Kinematics parameters and natural modes.

By combining Eqs. (5.3), (5.4) and (5.6), we obtain

$$\Phi_b = \frac{1}{2}\begin{Bmatrix} \phi_a \\ \phi_s \\ \phi_e \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} k_a & \cdot & \cdot \\ \cdot & k_s & \cdot \\ \cdot & \cdot & k_e \end{bmatrix} \begin{Bmatrix} \phi_a \\ \phi_s \\ \phi_e \end{Bmatrix}, \tag{5.7}$$

where

$$k_a = EA\ell, \quad k_s = \frac{4EJ}{\ell}, \quad k_e = \frac{12EJ}{\ell(1+\beta)}, \quad \beta := \frac{12EJ}{GA^*\ell^2}.$$

By simple kinematical considerations we can relate the natural strains to the element displacement vector (Fig. 3)

$$\boldsymbol{u}_b := \{U_i, W_i, \varphi_i, U_j, W_j, \varphi_j\}^{\mathrm{T}} \tag{5.8}$$

collecting displacements and rotations in the end-sections of the beam, with reference to a global Cartesian system X, Y. We obtain

$$\begin{Bmatrix} \phi_a \\ \phi_s \\ \phi_e \end{Bmatrix} := \boldsymbol{A}_b\boldsymbol{u}_b, \quad \boldsymbol{A}_b := \begin{bmatrix} -c/\ell & -s/\ell & 0 & c/\ell & s/\ell & 0 \\ 0 & 0 & 1/2 & 0 & 0 & -1/2 \\ -s/\ell & c/\ell & 1/2 & s/\ell & -c/\ell & 1/2 \end{bmatrix} \quad \begin{matrix} s = \sin\alpha, \\ c = \cos\alpha, \end{matrix} \tag{5.9}$$

α being the angle between the two reference systems. Therefore, by the definition

$$\Phi_b := \tfrac{1}{2}\boldsymbol{u}_b^{\mathrm{T}}\boldsymbol{K}_b\boldsymbol{u}_b \equiv \boldsymbol{u}_b^{\mathrm{T}}\boldsymbol{s}_b, \tag{5.10}$$

the element elastic stiffness matrix and the element internal forces vector can be expressed as:

$$\boldsymbol{K}_b := \boldsymbol{A}_b^{\mathrm{T}} \begin{bmatrix} k_a & \cdot & \cdot \\ \cdot & k_s & \cdot \\ \cdot & \cdot & k_e \end{bmatrix} \boldsymbol{A}_b, \qquad \boldsymbol{s}_b := \boldsymbol{A}_b^{\mathrm{T}} \begin{Bmatrix} m_a \\ m_s \\ m_e \end{Bmatrix}. \tag{5.11}$$

The elastic stiffness matrix $\boldsymbol{K}_e$ and the internal force vector $\boldsymbol{s}$ for the overall frame are obtained by standard assemblage of all beam element contributions:

$$\boldsymbol{K}_e := \sum_b \mathscr{A}(\boldsymbol{K}_b), \qquad \boldsymbol{s} := \sum_b \mathscr{A}(\boldsymbol{s}_b).$$

5.2. Admissible domain

To simplify the analysis, we assume that the plastic admissibility condition for the beam element can be expressed in terms of the bending moment M alone:

$$M_y^-[x] \leqslant M[x] \leqslant M_y^+[x],$$

$M_y^-[x]$ and $M_y^+[x]$ being the negative and positive yield bending values. With these assumptions, the elastic domain becomes the line segment

$$\mathbb{E}[x] := \left\{ M[x] : M_y^-[x] \leqslant M[x] \leqslant M_y^+[x] \right\}.$$

Denoting with $M_i^{\mathrm{e}}[x]$ the elastic solutions due to the basic external loads $\boldsymbol{p}_i$, the elastic stress domain is defined by

$$\mathbb{S}_e[x] := \left\{ M^{\mathrm{e}}[x] : M^{\mathrm{e}}[x] = \sum_{i=1}^{p} \alpha_i M_i^{\mathrm{e}}[x],\ \alpha_i^{\min} \leqslant \alpha_i \leqslant \alpha_i^{\max} \right\}.$$

Then, by taking

$$M_e^-[x] := \min(M^{\mathrm{e}}[x]), \quad M_e^+[x] := \max(M^{\mathrm{e}}[x]), \qquad \forall M^{\mathrm{e}}[x] \in \mathbb{S}_e, \tag{5.12}$$

the admissible shakedown domain can be defined, according to Eq. (4.25), as

$$\mathbb{E}_s[\lambda] := \{\overline{M}[x] : M_y^-[x] - \lambda M_e^-[x] \leqslant M[x] \leqslant M_y^+[x] - \lambda M_e^+[x],\ \forall M^{\mathrm{e}}[x] \in \mathbb{S}[x]\}. \tag{5.13}$$

To implement condition $M \in \mathbb{E}_s[\lambda]$ continuously in the entire interval $x \in [0 \cdots \ell]$ can be numerically cumbersome. We choose here to require condition (5.13) to be satisfied only in the two cross-sections i and j, that is we assume the shakedown beam-element admissible domain $\mathbb{E}_{\mathrm{sb}}[\lambda]$ be expressed by the conditions

$$M_{yi}^- - \lambda M_{ei}^- \leqslant M_i \leqslant M_{yi}^+ - \lambda M_{ei}^+, \qquad M_{yj}^- - \lambda M_{ej}^- \leqslant M_j \leqslant M_{yj}^+ - \lambda M_{ej}^+, \tag{5.14}$$

where due to Eq. (5.3),

$$M_i = (m_s + m_e)/2, \qquad M_j = (m_s - m_e)/2.$$

From Eq. (5.14) we can easily compute the largest admissible load amplifier $\bar{\lambda}$. We obtain

$$\bar{\lambda} = \min \left\{ \frac{M_{yk}^+ - M_{yk}^-}{M_{ek}^+ - M_{ek}^-} \right\}, \quad \forall k,$$

where k relates to the control end-sections of all beams of the frame.

It is worth remembering that Eq. (5.14) only represents a very simple, approximate, discrete implementation of condition (5.13), that however can be considered adequate for our purposes, the discretization error being reduced by using sufficiently small elements. Other, even more convenient, choices are possible, for instance that of using three discrete conditions, including the mid section. The convenience of the present choice lies in a numerically straightforward implementation of the return mapping algorithm.

5.3. Return mapping for the beam element

Return mapping scheme (3.10) can be defined by expressing the Haar–Kármán condition (3.11) for the whole element, that is

$$\Phi_b := \frac{1}{2} \begin{Bmatrix} m_s - m_s^* \\ m_e - m_e^* \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1/k_s & \cdot \\ \cdot & 1/k_e \end{bmatrix} \begin{Bmatrix} m_s - m_s^* \\ m_e - m_e^* \end{Bmatrix} = \min, \quad \{m_s, m_e\} \in \mathbb{E}_{\mathrm{sb}}[\lambda], \tag{5.15}$$

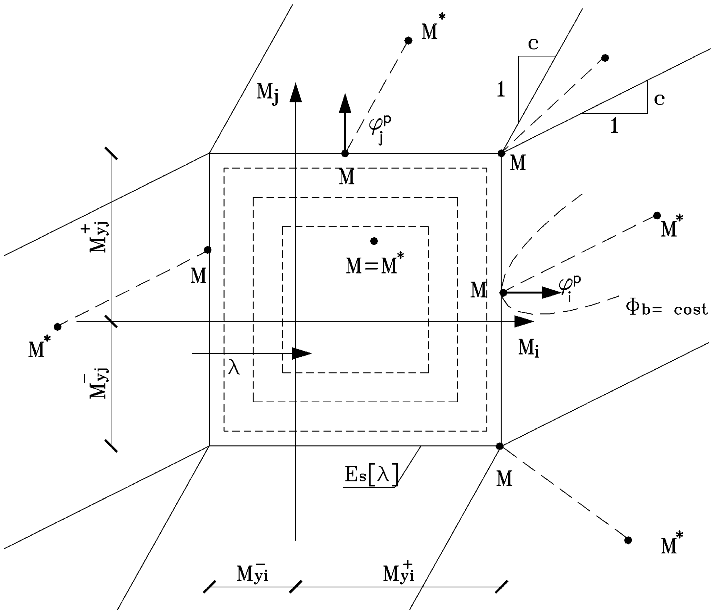

Fig. 4. Haar–Kármán problem for the beam element.

where

$$m_s^* = m_{s0} + k_s(\phi_{sj} - \phi_{s0}), \qquad m_e^* = m_{e0} + k_e(\phi_{ej} - \phi_{s0})$$

is the elastic estimate corresponding to $\boldsymbol{u}_j$. By referring to the end-section moments M_i, M_j and M_i^*, M_j^*, we obtain (Fig. 4):

$$\Phi_b = \frac{k_e + k_s}{k_s k_e} \begin{Bmatrix} M_i - M_i^* \\ M_j - M_j^* \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} \begin{Bmatrix} M_i - M_i^* \\ M_j - M_j^* \end{Bmatrix}, \quad c = \frac{k_e - k_s}{k_e + k_s}. \tag{5.16}$$

So, the solution of problem (5.14) and (5.15) can be geometrically defined as in (5.17). Excluding banal cases, it is then characterized by

$$\begin{cases} (M_i^* - M_i) + c(M_j^* - M_j) = 0 & \text{if } \; M_j = M_{yj}^+ - \lambda M_{ej}^+ \; \text{ or } \; M_j = M_{yj}^- - \lambda M_{ej}^-, \\ (M_j^* - M_j) + c(M_i^* - M_i) = 0 & \text{if } \; M_i = M_{yi}^+ - \lambda M_{ei}^+ \; \text{ or } \; M_i = M_{yi}^- - \lambda M_{ei}^- \end{cases}$$

and can be simply obtained, as the reader could easily verify, by the following algorithmic sequence:

$$\begin{aligned} \widetilde{M}_i &:= \max\{M_{yi}^- - \lambda M_{ei}^-, \min\{M_i^*, M_{yi}^+ - \lambda M_{ei}^+\}\}, \\ M_j &:= \max\{M_{yj}^- - \lambda M_{ej}^-, \min\{M_i^* - c(M_j^* - \widetilde{M}_i), M_{yj}^+ - \lambda M_{ej}^+\}\}, \\ M_i &:= \max\{M_{yi}^- - \lambda M_{ei}^-, \min\{M_i^* - c(M_j^* - M_j), M_{yi}^+ - \lambda M_{ei}^+\}\}. \end{aligned} \tag{5.17}$$

Hence the solution in terms of natural stress is easily obtained:

$$m_s := M_i + M_j, \qquad m_e := M_i - M_j.$$

The process is synthesized in Table 2.

Table 2
Pseudo-code for the return mapping scheme

For each beam element:

1. Compute natural strains ϕ_s and ϕ_e due to $\boldsymbol{u}_j$ through Eq. (5.5).
2. Compute the elastic prediction m_s^* and m_e^*

$$m_s^* = m_{s0} + k_s(\phi_{sj} - \phi_{s0}), \qquad m_e^* = m_{e0} + k_e(\phi_{ej} - \phi_{s0}).$$

3. Perform the Haar–Kármán sequence

$$\begin{aligned} \widetilde{M}_i &:= \max\{M_{yi}^- - \lambda M_{ei}^-, \min\{M_i^*, M_{yi}^+ - \lambda M_{ei}^+\}\}, \\ M_j &:= \max\{M_{yj}^- - \lambda M_{ej}^-, \min\{M_i^* - c(M_j^* - \widetilde{M}_i), M_{yj}^+ - \lambda M_{ej}^+\}\}, \\ M_i &:= \max\{M_{yi}^- - \lambda M_{ei}^-, \min\{M_i^* - c(M_j^* - M_j), M_{yi}^+ - \lambda M_{ei}^+\}\}. \end{aligned}$$

4. Finally compute

$$m_s := M_i + M_j, \qquad m_e := M_i - M_j.$$

6. Numerical results

Numerical results related to some simple test cases are presented in this section in order to show the general behavior of the proposed solution method. All tests have been performed using an initial step-length factor $\beta_0 = 0.01$ and a reference loop number $\bar{n} = 6$. A relative tolerance factor $t_{\rm f} = 10^{-5}$ has been used for stopping the iteration scheme (4.11) by the condition

$$\|\boldsymbol{s}_j\| \leqslant t_{\rm f} \frac{\lambda_e}{p} \sum_i^{\rm p} (|\alpha_i^{\min}| + |\alpha_i^{\max}|)\|\boldsymbol{p}_i\|.$$

The same value has been assumed for stopping the incremental process by the condition

$$\frac{\lambda^{(k)} - \lambda^{(k-1)}}{\|\boldsymbol{u}^{(k)} - \boldsymbol{u}^{(k-1)}\|} < t_{\rm f} \frac{\lambda^{(k)}}{\|\boldsymbol{u}^{(k)}\|}, \qquad \|\boldsymbol{u}\| := \boldsymbol{u}^{\rm T}\boldsymbol{K}\boldsymbol{u}.$$

For all tests $M_y^+ = -M_y^- = M_y$ will be assumed.

6.1. A simple frame

The first numerical test presented refers to the frame in Fig. 5, where all beams have the same section. To simplify the comparison with the analytical solution the shear contribution to strain energy is neglected ($\beta = 0$). The loads domain is defined as follows:

$$p[t] := \alpha_1[t]P_1 + \alpha_2[t]P_2, \quad 0 \leqslant \alpha_1 \leqslant 1, \;\; 0 \leqslant \alpha_2 \leqslant 2,$$

P_1 and P_2 being the horizontal and vertical forces. The elastic stresses envelope for the most meaningful sections of the frames are reported in Table 3.

The frame is subdivided into four beam elements. The elastic multiplier is obtained as $\lambda_e = 8M_y/7\ell = 228.5729$ by the max negative bending moment in node 4, while the analytical value of the shakedown multiplier is:

$$\lambda_a = \frac{4M_y}{\ell} = 266.6667.$$

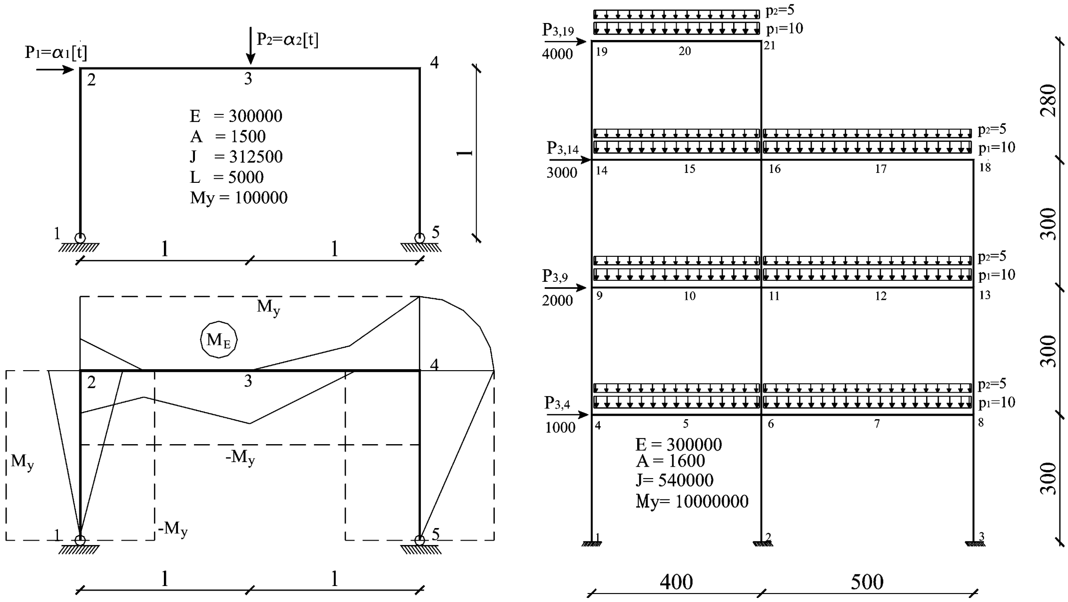

Fig. 5. Simple and four-floor frames.

Table 3
The M^+ and M^- function in some sections of the simple frame

M_2^-	M_2^+	M_3^-	M_3^+	M_4^-	M_4^+
−1875.0	2500	0	3125.0	−4375.0	0

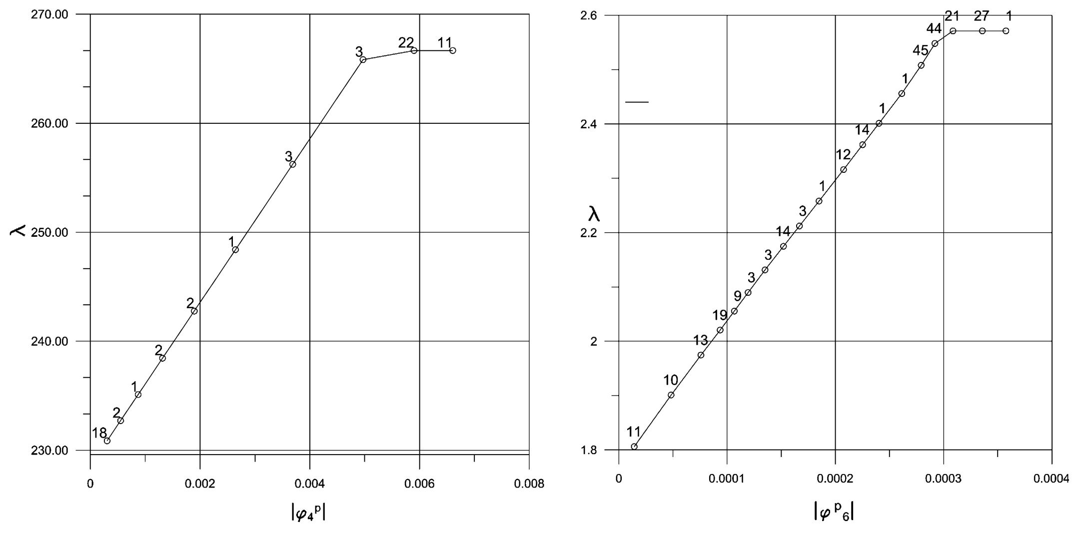

Fig. 6. Evolution of the analysis for the simple and four-floor frame.

The same values are obtained numerically. Fig. 6 reports the evolution of the shakedown multiplier versus plastic rotation of node 4 and also shows the number of iterations performed in each step. From a computational point of view the effort is, practically, the same as in elastic–plastic analysis using the same path-following formulation.

6.2. Four-floor frame

The second test-case refers to the simple four-floor frame under vertical and seismic forces shown in Fig. 5. The beam properties and the basic loads are also shown in the figure. The load domain is defined as follows:

$$\mathbf{p}[t] = \sum_{i=1}^{3} \alpha_i[t]\mathbf{p}_1, \quad 0.95 \leqslant \alpha_1[t] \leqslant 1.05, \;\; 0 \leqslant \alpha_2[t] \leqslant 1, \;\; -0.5 \leqslant \alpha_3[t] \leqslant 0.5.$$

One element is used to describe each column and two for each horizontal beam. The incremental process stopped after 19 steps (252 total loops) by providing the evaluation $\lambda_a = 2.57129$ for the shakedown safety factor (due to the presence of the distributed transversal load, this value, while accurate, is affected by some discretization error).

6.3. Large dimension frames

The advantages of the present method, when compared with other proposals, are obviously more and more evident as the number of elements increase. Table 5 reports some performance data referring to a series of test-cases relative to regular frames with different numbers of floors (n_f) and spans (n_s). A constant floor height $h_f = 300$ and a constant span length $l_s = 400$ is assumed for simplicity and 3 load conditions are assumed: two distributed vertical loads $p1 = 10$ and $p2 = 5$ and a seismic action defined as transversal floor forces linearly increasing by 500 from the ground to the top floor (see Fig. 7).

Beam properties are reported in Table 4 and the load domain is defined by

$$\mathbf{p}[t] = \sum_{i=1}^{3} \alpha_i[t]\mathbf{p}_1, \quad 0.9 \leqslant \alpha_1[t] \leqslant 1.0, \;\; 0 \leqslant \alpha_2[t] \leqslant 1, \;\; -1.0 \leqslant \alpha_3[t] \leqslant 1.0.$$

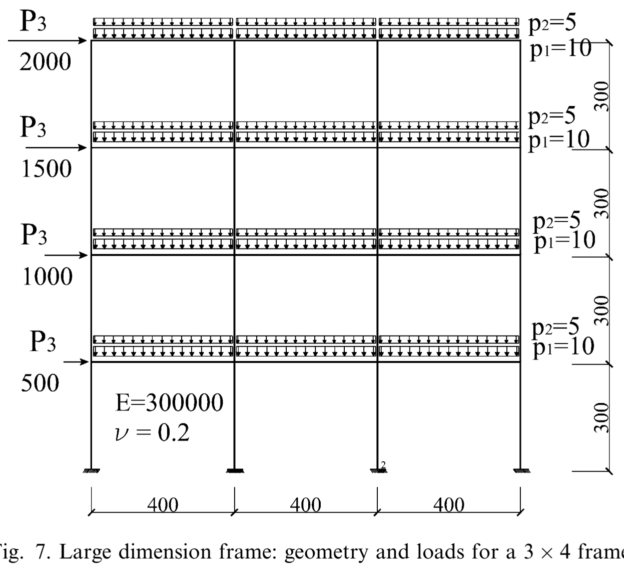

Fig. 7. Large dimension frame: geometry and loads for a 3 × 4 frame.

Table 4
Beams mechanical properties for large frames

	A	J	M_y
Vertical beams	1800	540,000	1,800,000
Horizontal beams	900	67,500	450,000

Table 5
Numerical performance of the proposed method for large frames

$n_s \times n_f$	λ_a	Steps	Loops
3×4	2.013382	32	240
4×6	1.399336	36	179
5×9	0.753276	30	140
6×10	0.720903	32	154

Table 6
Comparison between schemes (4.26) and (4.12) in limit analysis of large frames

$n_s \times n_f$	λ_e	λ_c	Scheme (4.26)		Scheme (4.12)	
			Steps	Loops	Steps	Loops
3×4	1.29336	2.46118	15	217	22	261
4×6	0.92763	1.86096	24	462	42	524
5×9	0.58349	1.20000	56	734	74	779
6×10	0.56268	1.15325	69	937	87	963

The results of the analysis are shown in Table 5 which reports the total number of steps and loops, the computational times required by the assemblage and decomposition of the stiffness matrix and that required by the incremental process. Times are expressed in milliseconds of Pentium II-300 Pc.

Note that the total number of loops is approximately constant (it actually depends on the tolerance assumed and, in some unpredictable manner, on the problem). So the time required by the incremental process is directly related to the dimension of the problems and tends to increase more slowly than that required by the decomposition of the elastic matrix. Computational times of 721 and 35 ms (Pentium II 300) have been required for the incremental process and the matrix decomposition in the 6×10 frame with a ratio of about 20. However this can be considered a case of relatively small dimensions (117 D.O.F.) and we can expect much smaller ratios for very large problems (thousands of D.O.F.) we can obtain by finite element discretizations of two- and three-dimensional continua.

6.4. Limit analysis problems

In Table 6, the numerical performances of the proposed scheme (4.12) are compared with that of a standard path-following scheme (4.26) for limit analysis problems. The analysis has been performed for the same test cases discussed in the previous section but assuming a fixed load combination defined by $\alpha_i = 1.0, \forall i$.

Note that the performances of the two schemes, in terms of both steps and number of loops, are almost the same, as expected (the standard scheme, which does not require the computation of vector $\boldsymbol{y}_j$, is however more convenient in this case).

7. Conclusions

A formulation of the shakedown problem, suitable for performing numerical analyses, has been presented. An incremental-iterative solution method has been proposed which is able to provide the shakedown solution in a general FEM context. Its convergence features have been discussed and proved and, as an example, its implementation has been detailed for the simple context of plane frames.

The proposed strategy appears to be more efficient than other numerical methods proposed in literature, especially for large dimension problems where the matrix operation prevail. In fact, it requires essentially:

1. The assemblage and the triangular decomposition of the elastic stiffness matrix $\boldsymbol{K}_e$ once and for all. This is a standard process, also needed for the elastic analysis, which requires about $nm^2/2$ basic floating-point operations, n being the dimension of the matrix and m its half-band size.
2. The elastic solution $\boldsymbol{u}_i = \boldsymbol{K}_e^{-1}\boldsymbol{p}_i$ for each of the basic loads $\boldsymbol{p}_i$. This is also a standard process and requires about $2pnm$ floating-point operations, p being the number of basic loads.
3. A series of iteration loops, each performing an evaluation of $\boldsymbol{s}_j = \boldsymbol{s}[\boldsymbol{u}_j, \lambda_j]$ and $\boldsymbol{y}_j = \boldsymbol{y}[\boldsymbol{u}_j, \lambda_j, \lambda_j]$ for given $\boldsymbol{u}_j$ and λ_j and the solutions $\dot{\boldsymbol{u}} = -\boldsymbol{K}_e^{-1}(\boldsymbol{s}_j + (\lambda_{j+1} - \lambda_j)\boldsymbol{y}_j)$. This requires about $4nm$ floating-point operations for each loop.

Denoting with p, s and l the number of basic loads, steps and average loops per single step, the entire shakedown solution process then requires about

$$\tfrac{1}{2}nm^2 + (2p + 4sl)nm$$

floating-point operations. We experienced $s \approx 30$ and $l \approx 5$ for our frame tests and can expect the same or even smaller values in large finite element discretizations of bi-dimensional problems where the evolution of the plastic process could be more regular. Therefore, when considering very large structures, that is for large m, the computational burden of shakedown analysis should become comparable with that needed by a simple elastic solution which will require $nm^2/2 + 2pnm$ operations.

In this way it is apparent that the proposed method is much more efficient than direct methods based on standard optimization algorithms. It is also more efficient than iterative formulations based on the so-called elastic compensation method of Ponter [32,33]. The latter in fact develops into a sequence of linear analyses each of which requires a complete assemblage and decomposition of a pseudo-elastic matrix (so about $nm^2/2 + 2nm$ for each step). Furthermore its convergence is relatively slow, so the number of successive analyses can be large (several tens, according to [33]). The computational advantages are evident, particularly for large structures.

The proposed method has real minimal implementational differences with respect to the standard path-following algorithms currently used for evaluating the equilibrium path of an elastic–plastic structure. This fact should make it very easy to modify commercial codes aimed at the elastic–plastic analysis into codes able to perform shakedown analysis in technically relevant applications.

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